Three hours

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

M.Sc. in Informatics

Foundations of Computing

Monday 26th January 2009

Time: 14:00 – 17:00

Please answer THREE questions from the FIVE questions provided

The use of electronic calculators is not permitted
1. a) Describe the six categories of symbols occurring in the (first-order) special notation of set theory. (7 marks)

b) To illustrate your answer to (a), consider the relation BrotherOf, such that ‘x BrotherOf y’ means ‘x is a brother of y’, and which has the following outline definition:

“A person P is the brother of another person Q iff P is male and different from Q, and they have the same father and the same mother.”

i) First, convert this outline definition to an equivalent English one, such that the latter can be directly translated into the notation of your answer to (a). You should assume the concepts of father, mother and maleness are predefined and may be used.

ii) Then express your English definition in this special notation. (7 marks)

c) i) Give the formal definition of the operations $n \text{ DIV } d$ and $n \text{ MOD } d$, over the integer numbers. Explain how this may be slightly simplified when restricted to the natural numbers.

ii) On the basis of your answer to (i) explain exactly why the division of any number by 0 is impossible.

iii) From your answer to (i) derive the simplest description of the set of divisors of 0 in the form of an abstraction. (6 marks)
2. a) i) Give the notation for a couple and define this notion informally.
   
   ii) Give Kuratowski’s set-theoretical definition of a couple.
   
   iii) Briefly describe how, from a couple in Kuratowski’s form, the first component may be extracted, and then the second component. (Consider the two cases: (1) where the two components are equal and (2) where the components are different.)
   
   (4 marks)

b) i) Define the cartesian product \( A \times B \) of any two sets \( A \) and \( B \), informally and formally.

   ii) Enumerate \( A \times B \) for \( A = \{1, 2\} \) and \( B = \{a, b, c\} \).

   iii) For any sets \( A \) and \( B \), establish the law that determines the cardinality of \( A \times B \) from the cardinalities of its operands \( A \) and \( B \).

   (4 marks)

c) i) Give the definition of a \( n \)-ary function \( F \).

   ii) Give two examples of binary (2-ary) functions, one from number theory and one from set theory.

   (4 marks)

For parts (d) and (e), consider the game of chess.

d) i) Let \( \text{Ranks} \) be the set of natural numbers from 1 to 8 inclusive, and \( \text{Files} \) the set of letters from a to h. In chess, ranks (rows) and files (columns) are indexed by these two sets respectively. Define \( \text{Ranks} \) by an abstraction and \( \text{Files} \) by an enumeration in alphabetical order. Then describe the chess board as a cartesian product of these two sets, ignoring the colour (black or white) of its component squares.

   ii) Introduce the colour of the component squares as a function \( \text{Clr} \) with domain your answer to (i) and codomain the set \{White, Black\}, without fixing the graph of \( \text{Clr} \) yet.

   (4 marks)

e) i) Introduce a new set \( \text{NumFiles} \) defined like \( \text{Ranks} \). Then using the successor function \( \text{Suc} \) sending each \( x \): \( \text{Files} \) to its successor in alphabetical order (i.e. \( \text{Suc}(a) = b \) etc.), formally define the natural bijection \( B \) from \( \text{Files} \) to \( \text{NumFiles} \) sending a to 1, b to 2, etc.

   ii) Given that the top left corner \((a, 8)\) of a chess board is White, find a simple formula that defines \( \text{Clr}(x) \) for each \( x \): \( \text{Dom}(\text{Clr}) \) using the bijection \( B \). Hint: send \((a, 8)\) to \((1, 8)\), and then to a simple numeric function of \((1, 8)\), etc.

   (4 marks)
3. A systems analyst is to build a set-theoretical model of a firm. The analyst has identified three primitive sets and relations on these sets, which are given below informally.

a) The primitive sets are as follows:

<table>
<thead>
<tr>
<th>Set</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depts</td>
<td>The set of departments of the firm</td>
</tr>
<tr>
<td>Managers</td>
<td>The set of managers of the firm</td>
</tr>
<tr>
<td>Workers</td>
<td>The set of workers of the firm</td>
</tr>
</tbody>
</table>

i) For any set $S$ (of sets), the unary relation $\text{IsDisjPw}(S)$ expresses that the members of $S$ are pairwise disjoint. Define $\text{IsDisjPw}$ formally.

ii) Express the fact that the above three sets are pairwise disjoint, using $\text{IsDisjPw}$.

(4 marks)

b) The relations (some of which functions) are as follows. Each is named and described informally. Introduce each relation in the appropriate notation, stating in particular its type, domain and codomain; briefly explain the notation; and formally express the companion constraints.

i) $Head$ Each department has just one head, who is a manager.

ii) $WDept$ Each worker belongs to just one department, the worker’s department (to be referred to directly).

iii) Every department is defined as the set of its workers, and has at least one member.

iv) $Boss$ Each worker has (at least) one boss, who is a manager. (We shall also say that a worker reports to his/her boss.)

v) $A worker’s boss is the head of the worker’s department.$

vi) $CoWkr$ Any two workers may be co-workers.

vii) Any two workers are co-workers iff they are different and belong to the same department.

(8 marks)

(Question 3 continues on the following page)
(Question 3 continues from the previous page)

c) Formally express the following statements. In each case indicate whether the statement is implied or not by previous statements and if implied, prove the implication.

i) No manager belongs to a department.

ii) If two workers are co-workers then they have the same boss.

iii) Any two different departments are disjoint.

iv) If Albert and Charles report to Beth and are not co-workers, then Beth heads more than one department.

(8 marks)
4. a) Summarise and critically assess the principles of Feature Notation, covering the following points:

i) The rules for the description of a class of objects by a Feature Notation format.

ii) The difference between variable features and fixed features (also known as invariants).

iii) The difference between primary features and secondary features.

(6 marks)

b) Consider the following definitions leading to that of a ‘club’ in Feature Notation:

\[
\begin{align*}
\text{Persons: Set} & \quad \text{Set of all persons} \\
\text{Stools: Set} & \quad \text{Set of all stools} \\
\text{IsDisjoint(People, Stools)} & \quad \text{Persons} \cap \text{Stools} = \emptyset \\
\text{C: Club} & \\
\text{CMems} & \subseteq \text{d} \text{ Persons} \quad \text{Membership of } C \\
\text{CStools} & \subseteq \text{d} \text{ Stools} \quad \text{Set of stools of } C \\
\text{CSt}: \quad \text{CMems} \rightarrow \text{CStools} & \\
\text{CSt(p)} & \quad \text{Stool owned by } p \quad (p: \text{CMems}) \\
\text{CCond1} & \quad \text{IsInjective (CSt)} \quad \text{CSt is injective}
\end{align*}
\]

Explain this definition in detail. In particular, justify the invariant \(\text{CCond1}\).

(8 marks)

c) Formally establish the following operations on Club, taking care to specify appropriate arguments, preconditions and results in each case:

i) An operation \(\text{MIns}\) which, given a club state \(C\) and a person \(p\), inserts \(p\) as a new member with no stool (‘standing member’)

ii) An operation \(\text{MRem}\) which, given a club state \(C\) and a member \(m\), removes \(m\) from the club, assuming \(m\) is a ‘standing member’

iii) An operation \(\text{MRem1}\) like \(\text{MRem}\) but without the assumption that \(m\) is a ‘standing member’

(6 marks)
5.  a) Draw the parse tree of the following wff:

\[ \alpha : \ (p \land \neg q) \iff p \]

(4 marks)

b) i) Define the concept of a tautology.

ii) Determine by a truth table whether \( \alpha \) is a tautology, a contradiction, or neither of those things. Note: a contradiction is any wff \( \beta \) such that \( \neg \beta \) is a tautology. The first two columns of your truth table must be

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

(5 marks)

c) Explain what each entry of your truth table represents, in terms of the semantic function \( M \). In particular for each entry of the first row, formally describe in detail the rule by which the entry is computed.  

(6 marks)

d) Consider the following deduction:

- If the sky is clear then Backus can spot Venus.
- Backus cannot spot Venus.
- Therefore:
- The sky is cloudy.

Making any appropriate assumption(s), formulate this deduction as a tautological implication and prove it, using contraposition.  

(5 marks)