Three hours

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

M.Sc. in Informatics

Foundations of Computing

Date: Friday 29th January 2010
Time: 14.00 – 17.00

Please answer THREE questions from the FIVE questions provided

The use of electronic calculators is NOT permitted
1. a) i) Give the formal definition of $\text{Pow}(A)$, i.e. the powerset of any set $A$.

ii) Enumerate the set $\text{Pow}(A)$ for the following subsets $A_i \subseteq \text{Nat}$:

\[
\begin{align*}
A_0 &= \emptyset \\
A_1 &= \{1\} \\
A_2 &= \{1, 2\} \\
A_3 &= \{1, 2, 3\}
\end{align*}
\]

(4 marks)

b) Give an exact definition of $q = n \text{ DIV } d$ and $r = n \text{ MOD } d$, for any two natural numbers $n$ and $d$. Explain exactly why your definition implies that $q$ and $r$ are not defined if $d = 0$, and comment on the case $n = 0$. (5 marks)

c) Give the notation and formal definition of the following set-theoretical operations:

i) The union of two sets $A$ and $B$

ii) The intersection of two sets $A$ and $B$

iii) The relative complement of $A$ and $B$

iv) The union of a set $A$ of sets (‘generalized union’)

v) The intersection of a set $A$ of sets (‘generalized intersection’)

For which set $A$ is the last operation not defined and why? (5 marks)

d) Two business firms are planning to merge. A consequence would be that their respective lists of customers would be combined, an operation similar to the merger of the lists of pen-friends of two acquaintances.

i) Briefly describe the set-theoretical definition of the merger of two lists of pen-friends, any attendant difficulties, and how these can be resolved formally.

ii) If $n$ pen-friends decide to merge their lists, for any $n \geq 2$, how should the model be generalized?

iii) Discuss how the pen-friend merger model can be applied to the merger of the firms’ customer lists, with due emphasis on the differences between these two problems and how the original model would have to be adapted accordingly. (6 marks)
2.  a) i) Give an informal definition of a function, and the two set-theoretical definitions of this concept.

ii) Define the concept of a bijection.

iii) Consider the following two sets: \( A = \{0, 1, 2, 3, 4\} \) and \( B = \{1, 3, 5, 7, 9\} \). Construct a bijection \( f: A \rightarrow B \) (by specifying \( f(x) \) for each \( x \in A \)) and from the existence of this bijection, draw an appropriate conclusion regarding the cardinalities of \( A \) and \( B \). (6 marks)

b) Outline four different non-arithmetic examples of functions (other than the ‘rules of chess’). In each application, indicate the domain and codomain of the function and suggest possible alternatives, if appropriate. Discuss any special property that may normally be expected to hold in each case. (8 marks)

c) i) Describe a configuration of the game of chess as a (partial) function from the chessboard (modelled as a cartesian product) to the set of piece types of the game.

ii) Describe the rules of chess as a (possibly partial) function giving for each configuration a set of allowed moves leading to the next configuration. Is this function partial or total, and why? (6 marks)
3. A systems analyst sets out to build a set-theoretical model of an international seminar. The analyst identifies various primitive sets and relations on these sets, which are described informally below. You are to express these relations formally, and to prove some properties in outline.

a) There are three primitive sets, given with a brief justification in parentheses:

- **Persons** Set of all possible persons. (Potential participants.)
- **Languages** Set of all possible languages. (Participants’ languages determine their ability to communicate with each other.)
- **Countries** Set of all possible countries. (Each participant belongs to and possibly represents one country.)

i) For any set $S$ (of sets), the unary relation $\text{IsDisjPw}(S)$ expresses that the members of $S$ are pairwise disjoint. Define $\text{IsDisjPw}$ formally.

ii) Express the fact that the above three sets are pairwise disjoint, using $\text{IsDisjPw}$. (4 marks)

b) The analyst identifies further relations between the primitive sets. The names of these relations are given below, together with an informal description of what they are meant to model, and some associated constraints. Introduce each relation in the appropriate notation, stating in particular its domain and codomain and any special property it may have; express any additional constraint; and briefly explain how the notation renders the original informal description. Note some of the relations are functions, and this must be allowed for in the notation.

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
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<tbody>
<tr>
<td>i)</td>
<td><strong>Nati</strong> Each person has one nationality, i.e. the country to which he/she belongs.</td>
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<tr>
<td>ii)</td>
<td><strong>CLans</strong> Each country speaks a finite and nonempty set of languages. (For each country, we want to refer to the associated set of languages.)</td>
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<tr>
<td>iii)</td>
<td><strong>PLans</strong> Each person speaks a finite and nonempty set of languages. (For each person, we want to refer to the associated set of languages.) Any person only speaks languages of their country.</td>
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<tr>
<td>iv)</td>
<td><strong>Com</strong> Any two persons may or may not be able to communicate directly. They can do so iff they speak a common language. (8 marks)</td>
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(Question 3 continues on the following page)
c) Formalise the following sentences. (For a person, ‘to be French’ means ‘to belong to France’ etc.)

S0: The only languages spoken in France are French and German.
S1: Any French person speaks French.
S2: Pierre is French and speaks only one language.
S3: Pierre does not speak German.
S4: German is the only language spoken in Germany.
S5: Helga is German.
S6: Pierre and Helga cannot communicate directly.  (8 marks)
A systems analyst has been asked to design a discussion group website. He has established a first semi-formal model of the group’s application domain, as a collection of sets, relations and functions. Your task is to formalize this model fully, in set-theoretical notation.

a) There are two primitive sets: **Persons** and **Strings**. The former represents the set of all possible persons; the latter, the set of strings of characters such as headings, statements, etc.

i) Introduce these two sets, simply just as sets.

ii) Formally express the property that these two sets are disjoint. (2 marks)

b) Formally specify the following additional sets from the informal descriptions given:

i) **Members** Set of members of the discussion group. Each member is a person.

ii) **Topics** Set of topics debated. Each topic is a string.

iii) **Statements** Set of statements made by members. Each statement is a string.

iv) **Views** Set of views of members on topics. Each view consists of a statement and the name of the member who made the statement.

v) **Positions** Set of positions any member may adopt with respect to any view. For any view a member may adopt one of three positions: to accept it, to reject it, or to have no view. (Give each position an appropriate name.) (6 marks)

c) The remaining elements of the model are relations, functions and associated constraints. For each item, first introduce it formally with a brief explanation. Then specify the item formally when this is required, from the informal description.

i) **ViewsOnTopics** For any topic, there is a set of views expressed on it. It must be possible to refer to this set.

ii) **Pos** For any member, any topic and any view on the topic, the member has a position on this view, one of the three available options.

iii) **Concur** For any two members, any topic and any view on the topic, the two members may concur or not.

iv) Two members concur on a view (about any topic) iff they have the same position on this view.

v) **ConcurOnTopic** For any two members and any topic, the two members may concur or not on all the views about the topic.

vi) Two members concur on a topic iff they have the same position on each view expressed on the topic. (12 marks)
5. a) Draw the parse tree of the following wff:

\[ \alpha (p \Rightarrow q) \Leftrightarrow (p \lor q) \]  

(5 marks)

b) Give the truth tables of \( \lor \) and \( \Rightarrow \), in each case with the first two columns of the table as follows:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
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<tbody>
<tr>
<td>T</td>
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Note: you are allowed to combine the two tables.  

(4 marks)

c) i) Define the concept of a truth assignment and indicate how a truth assignment \( m \) is introduced as a function (i.e. using the arrow notation specifying its name, domain and codomain).

ii) Introduce the semantic function \( M \) and briefly describe its domain and codomain.

iii) For any wff \( \gamma \), explain what \( M(\gamma) \) represents and how this depends on a truth assignment, but do not specify \( M \) any further. Illustrate your answer by reference to your answer to (b).  

(5 marks)

d) i) Define the concept of a tautology \( \gamma \) with reference to your answer to (c).

ii) Prove that \( \alpha \) of (a) is a tautology by the truth table method with the first two columns of the table as in (b).

iii) Write a tautology \( \beta \) expressing \( p \Leftrightarrow q \) in terms of \( \Rightarrow \) and \( \land \). Do not prove it.  

(6 marks)

END OF EXAMINATION