Please answer any THREE questions from the FIVE questions provided

For full marks your answers should be concise as well as accurate.
Marks will be awarded for reasoning and method as well as being correct.

This is a CLOSED book examination

The use of electronic calculators is NOT permitted
1. a) Define the six categories of symbols occurring in the (first-order) special notation of set theory. (9 marks)

b) To illustrate your answer to part (a), define a relation SisterOf, such that ‘\(x\) SisterOf \(y\)’ means ‘\(x\) is a sister of \(y\)’, by formalizing the following outline definition:

“A person \(P\) is the sister of another person \(Q\) iff \(P\) is female and different from \(Q\), and they have the same father and the same mother.”

(Thus the definition is to be derived from the concepts of father, mother and femaleness assumed predefined.)

i) First, convert this definition to a more precise English one, such that the latter can be directly translated into the notation of your answer to part (a).

ii) Then express your English definition in set-theoretic notation. (6 marks)

c) Repeat part (b) for a relation SiblingOf, allowing any \(x\) to be the sibling of any \(y\) if they have at least one common parent. (5 marks)
2. a) i) Give the notation for a couple and define this notion informally.
ii) Give Kuratowski’s set-theoretical definition of a couple.
iii) State the fundamental property which defines when two couples are equal. 

(6 marks)

b) i) Define the cartesian product $A \times B$ of two sets $A$ and $B$, informally and formally.
ii) Enumerate $A \times B$ for $A = \{1, 2\}$ and $B = \{a, b, c\}$.
iii) For any set $B$, there is an obvious bijection $F$ from $\{1\} \times B$ to $B$. Define this bijection and list its graph for $B = \{a, b, c\}$ in tabular form. 

(7 marks)

c) Give all the definitions and notations of the following concepts:
i) Give an informal definition and the two formal definitions of a relation $R$.
ii) Define a function as a special case of a relation.
iii) For any relation $R$, we can construct the function $F$ that sends each $x$: $\text{Dom}(R)$ to the set of elements $y$ such that $x R y$. Introduce and define $F$ formally. Thence one may conclude that the two concepts of function and relation are essentially equivalent; clarify and justify this conclusion.

(7 marks)
3. An analyst sets out to build a set-theoretical model of an international conference. The analyst identifies various primitive sets and relations on these sets, which are described informally below. You are to express these relations formally, and to prove some properties in outline.

a) There are three primitive sets, given with a brief justification in parentheses:

<table>
<thead>
<tr>
<th>Set</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Persons</td>
<td>Set of all possible persons. (Potential participants.)</td>
</tr>
<tr>
<td>Languages</td>
<td>Set of all possible languages. (Participants’ languages determine their ability to communicate with each other.)</td>
</tr>
<tr>
<td>Countries</td>
<td>Set of all possible countries. (Each participant belongs to and possibly represents one country.)</td>
</tr>
</tbody>
</table>

i) For any set \( S \) (of sets), the unary relation \( \text{IsDisjPw}(S) \) expresses that the members of \( S \) are pairwise disjoint. Define \( \text{IsDisjPw} \) formally.

ii) Express the fact that the above three sets are pairwise disjoint, using \( \text{IsDisjPw} \). (3 marks)

b) The analyst identifies further relations between the primitive sets. These are named below, together with informal descriptions of what they represent and some associated constraints. Introduce each relation in the appropriate notation, stating in particular its domain and codomain and any special property it may have; express any associated constraints; and add brief explanations. Note some of the relations are functions, and this must be allowed for in the notation.

<table>
<thead>
<tr>
<th>Relation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nati</td>
<td>Each person has one nationality, the country to which he/she belongs.</td>
</tr>
<tr>
<td>CLans</td>
<td>Each country speaks a finite and nonempty set of languages. (We want to refer to this set of languages.)</td>
</tr>
<tr>
<td>PLans</td>
<td>Each person speaks a finite and nonempty set of languages. (We want to refer to this set of languages.) Any person only speaks languages of their country.</td>
</tr>
<tr>
<td>Com</td>
<td>Any two persons may or may not be able to communicate. They can do so iff they speak a common language.</td>
</tr>
</tbody>
</table>

(8 marks)

c) Formalise the following sentences. (For a person, ‘to be French’ means ‘to belong to France’ etc.)

<table>
<thead>
<tr>
<th>Sentence</th>
<th>Formalisation</th>
</tr>
</thead>
<tbody>
<tr>
<td>i) S0:</td>
<td>The only languages spoken in France are French and German.</td>
</tr>
<tr>
<td>ii) S1:</td>
<td>Any French person speaks French.</td>
</tr>
<tr>
<td>iii) S2:</td>
<td>Pierre is French and speaks only one language.</td>
</tr>
<tr>
<td>vi) S3:</td>
<td>German is the only language spoken by a German.</td>
</tr>
<tr>
<td>v) S4:</td>
<td>Helga is German.</td>
</tr>
<tr>
<td>vi) S5:</td>
<td>Pierre and Helga cannot communicate.</td>
</tr>
</tbody>
</table>

(9 marks)
4. a) Summarise and critically assess the principles of Feature Notation, covering the following points:

i) Rules for the description of a class of objects by a Feature Notation format

ii) The difference between variable features and fixed features (a.k.a. invariants)

(6 marks)

b) Consider the following definitions leading to that of a ‘club’ in Feature Notation:

\[
\begin{align*}
\text{Persons: } & \text{Set} & \text{Set of all persons} \\
\text{Stools: } & \text{Set} & \text{Set of all stools} \\
\text{IsDisjoint(People, Stools): } & \text{Persons} \cap \text{Stools} = \emptyset
\end{align*}
\]

\[
\begin{align*}
C & \text{: Club} \\
CMems & \subseteq \text{Persons} & \text{Membership of } C \\
CStools & \subseteq \text{Stools} & \text{Set of stools of } C \\
CSt(p) & \rightarrow \text{CStools} & \text{Stool owned by } p \ (p: \text{CMems}) \\
CCond1 & \text{: IsInjective } (CSt) & \text{CSt is injective}
\end{align*}
\]

Explain this definition in detail. In particular, explain and justify the invariant \(CCond1\). (8 marks)

d) Formally establish the following operations on Club, taking care to specify appropriate arguments, preconditions and results in each case:

i) An operation \(MIns\) which, given a club state \(C\) and a person \(p\), inserts \(p\) as a new member with no stool (‘standing member’)

ii) An operation \(MRem\) which, given a club state \(C\) and a member \(m\), removes \(m\) from the club, assuming \(m\) is a ‘standing member’

iii) An operation \(MRem1\) like \(MRem\) but without the assumption that \(m\) is a ‘standing member’

(6 marks)
5.  

a) Draw the parse tree of the following wff:
\[ \alpha : \quad (p \lor q) \iff p \]

(4 marks)

b) i) Define the concept of a tautology.
ii) Determine by a truth table whether \( \alpha \) is a tautology, a contradiction, or neither of those things. Note: a contradiction is any wff \( \beta \) such that \( \neg \beta \) is a tautology. The first two columns of your truth table must be

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
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<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

(6 marks)

c) Explain what each entry of your truth table represents, in terms of the semantic function \( M \). In particular for each entry of the first row, describe the precise rule by which the entry is computed.

(5 marks)

d) Consider the following deduction:

i) If the network is on then Mark and Jane can exchange messages.
ii) Mark and Jane cannot exchange messages.
Therefore:
iii) The network is off.
Making any appropriate assumption(s), formulate this deduction as a tautological implication and prove it, using contraposition.

(5 marks)