Two hours

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

M.Sc. in Mathematics and Computational Science

Computational Finite Element Method

Date: Thursday 21st May 2009

Time: 14:00 – 16:00

Please answer any TWO Questions from the FOUR questions provided

This is an OPEN book examination

The use of electronic calculators is permitted provided they are not programmable and do not store text
1. a) Give the formulas for the Lagrange interpolation of a continuous function \( y = f(x) \) on an interval \([x_i, x_{i+1}]\subset \mathbb{R}\) by a quadratic polynomial \( P_2(x) \) defined locally on that interval. For this purpose, assume that there are three known function values \( f(x_i), f(x_{i+1/2}), \) and \( f(x_{i+1}) \), where \( x_{i+1/2} = (x_i + x_{i+1})/2 \). Give the explicit expressions for the Lagrange multipliers in this case, and explain the connection between the Lagrange multipliers and the local basis functions of a quadratic finite element. Discuss the connection between the local (element) basis functions and the global basis functions. (7 marks)

b) Consider the following two-point boundary value problem:

Find \( y \in C^2(0,1) \) satisfying:

\[
- y'' = 6e^{-x} \quad 0 < x < 1 \quad (1)
\]

subject to the boundary conditions

\[
y(0) = 6, \quad y(1) = 1 + 6/e. \quad (2)
\]

Derive the weak (variational) formulation of the problem and justify the choices of the continuous test and solution (trial) spaces in this context. Explain the difference between these two spaces in this case. (5 marks)

c) Suppose that the interval \([0,1]\) is subdivided into 2 finite elements of equal size \( h = 0.5 \). Derive the discrete weak formulation of the problem and give the choice of the discrete test and trial spaces. Explain in this context the difference between the conforming and the non-conforming approximation. (3 marks)

d) By adopting a piecewise quadratic basis set, show, in principle, how an elemental matrix is assembled and how the Dirichlet boundary conditions (2) can be imposed at the element matrix level. (5 marks)
Consider the following initial value problem:

Find $u(x, y, t) \in C^2(\Omega) \times C^1(T)$ satisfying:

$$\frac{\partial u}{\partial t} - \Delta u - k^2 u = f \quad \text{in } \Omega \times T = [0,1]^2 \times [0, \tau]$$  \hspace{1cm} (1)

where $k \in \mathbb{R}$, subject to homogeneous Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial \Omega,$$  \hspace{1cm} (2)

where $\partial \Omega$ is the boundary of $\Omega$ and the initial condition

$$u(x, y, 0) = 0 \quad \text{in } \overline{\Omega} = \Omega \cup \partial \Omega$$  \hspace{1cm} (3)

If the time discretisation is performed using the implicit Euler method (BDF-1), describe the finite element procedure for the spatial discretisation of the problem. In your answer address the following points:

a) Find the continuous weak formulation of the problem and describe the choices of the Sobolev spaces that are suitable candidates for the continuous test and trial spaces in this context. (4 marks)

b) Using finite dimensional approximations for the test and the trial spaces, derive the discrete weak formulation of the problem. (2 marks)

c) Suppose that the basis set of the finite-dimensional test and trial spaces consists of piecewise linear polynomials associated with a triangulation $\mathcal{T}_h$ of the domain $\Omega$. Describe the procedure for assembling the Galerkin system at each time step and give the formulas for the elements of the Galerkin matrix and the right-hand side vector. (14 marks)
3. Consider the biharmonic problem: Find \( u \in C^4(\Omega) \) satisfying

\[
\Delta^2 u = f \quad \text{in } \Omega \tag{1}
\]

where \( \Delta^2 \equiv \Delta(\Delta) \) is the biharmonic operator, subject to the following boundary conditions

\[
u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega, \tag{2}
\]

where \( \partial \Omega \) is the boundary of \( \Omega \). (Note that, as this is a fourth-order problem, for the uniqueness of the solution, two boundary conditions need to be satisfied simultaneously at each point of \( \partial \Omega \)). In order to facilitate the finite element approximation of the problem (1)-(2) by the Lagrangian elements, the differential equation (1) needs to be rewritten as a system of two lower-order problems. By introducing an auxiliary function \( v = \Delta u \), the problem (1) is equivalent to a system of two second-order problems: Find the pair \( (u, v) \in C^2(\Omega) \times C^2(\Omega) \) such that

\[
\begin{align*}
\Delta u &= v \\
\Delta v &= f
\end{align*} \tag{3}
\]

a) Derive the mixed finite element method for the solution of the problem (3), subject to boundary conditions (2). (Hint: Note the difference between the test and trial spaces for the unknown functions \( u \) and \( v \) – the former is constrained at the boundary, while the latter is not.) (8 marks)

b) Describe the choices of the finite element spaces that can be used to approximate the solutions \( u_h \) and \( v_h \) in the discrete weak formulation of the problem. (7 marks)

c) Give the block structure of the linear system obtained as a result of the finite element approximation. (5 marks)
4. a) Explain how the robustness of the fixed-point iterations can be improved in cases when they are applied to the solution of large, sparse linear systems obtained from the discretisations of second-order elliptic partial differential equations. Assume that a nested sequence of refined grids and the matrix representations of a continuous problem on these grids are given. Include in your answer the discussion on smooth and oscillatory components of a solution error and introduce the residual equation. (8 marks)

b) Based on the consideration from part a), define the V-cycle of multigrid (explaining all the components of the algorithm), and give a pseudo-code of a recursive definition of the V-cycle. (5 marks)

c) Explain how the transfer of information between different levels in MG hierarchy is accomplished. Assuming linear interpolation, explain a general procedure, based on full weighting, for assembling the interpolation matrix. Then assume that an equilateral triangle is refined into three smaller triangles by connecting its centroid with all three vertices. Give the elements of the interpolation matrix for this element patch (the matrix is of dimension 4x3). (7 marks)