Two hours

Examination Definition Sheet provided on pages 5-7

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Modal Logic and Description Logics

Date: Friday 20th May 2011
Time: 09:45 - 11:45

Please answer ALL three Questions

This is a CLOSED book examination
The use of electronic calculators is NOT permitted

[PTO]
**Answer all three questions**

1. a) Give a brief explanation of three of the following concepts. (6 marks)
   
   i) validity in modal logic
   ii) frame correspondence properties
   iii) axiom of positive introspection
   iv) temporal reading of \( \Box \) and \( \Diamond \)
   v) closed tableau

b) Consider the following variant of the model checking problem:

   **Input:** Given a finite Kripke model \( M \) and a modal formula \( \phi \)
   
   **Output:** true, if \( \phi \) is true at every world in \( M \); the worlds in which \( \phi \) is not true, otherwise

   Describe how the model checking algorithm that we discussed in class can be used as a subroutine to solve this variant of the model checking problem. (4 marks)

c) Use the semantic tableau method to show that the formula

\[
\Box (p \rightarrow q) \rightarrow \Box (\Box p \rightarrow \Box q)
\]

is valid in all transitive Kripke models. (10 marks)
2. a) Consider the following axiom $\phi$.

$$\Box \Diamond p \rightarrow p$$

i) Transform the negation of $\phi$ into negation normal form, and call the result $\psi$.

(2 marks)

ii) Translate $\psi$ into first-order logic.

(2 marks)

iii) Use the SOQE procedure to compute a correspondence property (if one exists) for the axiom $\phi$. Justify every step in your derivation. State what can be concluded.

(6 marks)

b) For each of the following statements, state whether they are true or false, and explain your decision in 1-2 sentences.

i) A TBox that is incoherent is also inconsistent.

(2 marks)

ii) Any decision procedure that decides, for $T$ an ALC TBox and $A$ an ALC ABox, whether $\langle T, A \rangle$ is consistent can also be used to decide whether $\langle T, A \rangle \models C \sqsubseteq D$ holds.

(2 marks)

iii) Let $T$ be an ALC TBox, $A$ an ALC ABox, and $\langle T, A \rangle$ consistent. If $T \cup A \models C \sqsubseteq D$, then $T \models C \sqsubseteq D$.

(2 marks)

iv) Whenever an ALC ontology has a model, it has infinitely many models.

(2 marks)

v) Let $O$ be an ALC ontology with $O \models \eta$, and $J \subseteq O$ a justification for $\eta$ in $O$ with $\alpha \in J$. Then $O \setminus \{\alpha\} \not\models \eta$.

(2 marks)
3. Consider the following TBox \( T \) and the following ABox \( A \):

\[
T = \begin{align*}
M &\equiv A \cap (\exists h. H) \cap (\exists h. M G), \\
B &\equiv A \cap (\exists h. F) \cap (\exists h. W), \\
FB &\equiv B \cap (\exists m. FL), \\
NFB &\equiv B \cap (\forall m. \neg FL), \\
T &\subseteq B \cap (\exists \ell. UK), \\
G &\subseteq FB \cap (\exists m. S), \\
SA &\subseteq FL
\end{align*}
\]

\[
A = \{ b1 : B \cap \exists \ell. UK, \\
b2 : B \cap \exists m. WA, \\
b3 : A \cap \exists h. B \cap \exists h. F \cap \exists m. SA, \\
(b3,w) : h, \\
w : W \}
\]

If you want, you can think that \( h \) stands for has part, \( m \) for moves, \( A \) for animal, \( B \) for bird, \( H \) for hair, \( F \) for feathers, \( W \) for Wings, \( FL \) for flying, \( SA \) for sailing, \( S \) for swimming.

a) Is \( T \) coherent? Is \( \langle T, A \rangle \) consistent? (4 marks)

b) Give an alternative version of Axiom 2 in \( \mathcal{ALC} \) that states that birds have two wings. (2 marks)

c) Using the terms used in \( T \) plus \( WA \) for walking and \( D \) for ducks, give an \( \mathcal{ALC} \) axiom that states that ducks are birds that move by swimming, flying, and walking, and nothing else. (2 marks)

d) Does \( T \) entail that birds and mammals are disjoint, i.e., does \( T \models B \subseteq \neg M \)? If yes, explain why. If not, give an axiom \( \alpha \) over \( H, F, h \) such that \( T \cup \{ \alpha \} \models B \subseteq \neg M \). (2 marks)

e) Translate Axiom 1 into an equivalent formula in first order logic. (2 marks)

f) For each of the following assertions \( \alpha \), determine whether \( \langle T, A \rangle \models \alpha \). If not, explain your answer. If yes, give a justification for this entailment.

   i) \( \alpha = b1 : T \) (2 marks)
   ii) \( \alpha = b2 : NFB \) (2 marks)
   iii) \( \alpha = b3 : FB \) (2 marks)

g) Explain the consequences of adding an axiom such as \( \ell \circ h \subseteq \ell \) to \( T \) (which can be read as “living in a part of an area implies living in that part”). (2 marks)
Examination definition sheet

Semantic tableau calculus for modal logic

Negation elimination

\[ \frac{w : \neg \neg \phi}{w : \phi} \quad \frac{w : \neg \top}{w : \bot} \]

\[ \alpha \text{-expansion} \]

\[ \frac{w : \alpha}{w : \alpha_1 \quad w : \alpha_2} \]

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
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<tbody>
<tr>
<td>$\phi \land \psi$</td>
<td>$\phi$</td>
<td>$\psi$</td>
</tr>
<tr>
<td>$\neg (\phi \lor \psi)$</td>
<td>$\neg \phi$</td>
<td>$\neg \psi$</td>
</tr>
<tr>
<td>$\neg (\phi \rightarrow \psi)$</td>
<td>$\phi$</td>
<td>$\neg \psi$</td>
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\[ \beta \text{-expansion} \]

\[ \frac{w : \beta}{w : \beta_1 \mid w : \beta_2} \]

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<td>$\phi$</td>
<td>$\psi$</td>
</tr>
<tr>
<td>$\phi \rightarrow \psi$</td>
<td>$\neg \phi$</td>
<td>$\psi$</td>
</tr>
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Closure

\[ \frac{w : \neg \phi}{w : \phi} \quad \frac{w : \bot}{w : \bot} \]

\[ \gamma \text{-expansion} \]

\[ \frac{w : \gamma}{(w,u) : R \quad u : \gamma_0} \]

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\gamma_0$</th>
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<tbody>
<tr>
<td>$\Box \phi$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>$\neg \Box \phi$</td>
<td>$\neg \phi$</td>
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\[ \delta \text{-expansion} \]

\[ \frac{w : \delta}{(w,u) : R \quad u : \delta_0} \]

where $u$ is a fresh constant

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<tr>
<td>$\Diamond \phi$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>$\neg \Box \phi$</td>
<td>$\neg \phi$</td>
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Reflexivity

\[ (w,w) : R \]

provided $w$ occurs in a labelled formula on the branch

Seriality

\[ (w,u) : R \]

provided $w$ occurs in a labelled formula on the branch and $u$ is a fresh constant

Symmetry

\[ (w,u) : R \quad (u,w) : R \]

Transitivity

\[ (w,u) : R \quad (u,v) : R \quad (w,v) : R \]

Euclideanness

\[ (w,u) : R \quad (w,v) : R \quad (u,v) : R \]
SOQE calculus

**Ackermann**

\[
\frac{N, \alpha_1(x) \lor P(x), \ldots, \alpha_n(x) \lor P(x)}{N(P(y))(\alpha_1 \land \ldots \land \alpha_n)(x)}
\]

provided

(i) \( P \) is the largest non-base symbol such that \( P \) does not occur in any of the \( \alpha_i \)

(ii) \( N \) is negative wrt \( P \)

(iii) \( N \) does not share any variables with any of the \( \alpha_i \) and each distinct pair of \( \alpha_i \) and \( \alpha_j \)
does not have any variables in common apart from possibly \( x \).

**Purify**

\[
\frac{N}{N(P(y))}
\]

provided \( P \) is the largest non-base symbol and (ii) above

**Eliminate \( \land \)**

\[
\frac{N, \alpha \land (\beta_1 \land \ldots \land \beta_m)}{N, \alpha \land \beta_1, \ldots, \alpha \land \beta_m}
\]

\[
\frac{N, \beta_1 \land \ldots \land \beta_m}{N, \beta_1, \ldots, \beta_m}
\]

provided the main premise is positive wrt the largest non-base symbol \( P \)

**Eliminate \( \forall \)**

\[
\frac{N, \alpha \lor \forall x \beta(x)}{N, \alpha \lor \beta(x)}
\]

\[
\frac{N, \beta(x)}{N, \beta(x)}
\]

provided

(i) the main premise is positive wrt the largest non-base symbol \( P \)

(ii) \( y \) is a fresh variable

**Eliminate \( \exists \)**

\[
\frac{N, \alpha \lor \exists x \beta(x)}{N, \beta(x)}
\]

\[
\frac{N, \exists x \beta(x)}{N, \beta(x)}
\]

provided

(i) the main premise is positive wrt the largest non-base symbol \( P \)

(ii) the main premise does not have any free variables

(iii) \( a \) is a fresh constant

**Term abstraction**

\[
\frac{N, \alpha \lor P(s)}{N, x \not\approx s \lor \alpha \lor P(x)}
\]

provided

(i) \( P \) is the largest non-base symbol in \( \alpha \lor P(s) \)

(ii) \( x \) is a fresh variable

**Variable renaming**

\[
\frac{N, \alpha(x)}{N, \alpha(x)}
\]

provided

(i) \( \alpha \) is positive wrt the largest non-base symbol

(ii) \( x \) occurs freely in \( \alpha \) and \( y \) does not occur in \( \alpha \)
Sign switching \[ N \xrightarrow{\text{simpl} (N^P_{\neg P(x)} \neg)} \]

provided

(i) \( N \) closed wrt other rules
(ii) \( P \) is a maximal non-base symbol in \( N \)
(iii) sign switching wrt \( P \) has not been applied before
(iv) \( \text{simpl} \) eliminates all occurrences of \( \neg \neg \)

Eliminate \( \not\approx \) \[ N, x \not\approx s \lor \alpha(x) \xrightarrow{N, \alpha(x/s)} \]

provided either \( \alpha \) does not contain a non-base symbol, or if it does, \( x \) is not an argument of the largest non-base symbol

Introduce \( \land \) \[ N, \alpha_1, \ldots, \alpha_n \xrightarrow{N, \alpha_1 \land \ldots \land \alpha_n} \]

provided

(i) none of the \( \alpha_i \) contains a non-base symbol
(ii) Eliminate \( \not\approx \) is not applicable to any of the \( \alpha_i \)

Introduce \( \forall \) \[ N, \alpha(x) \xrightarrow{N, \forall(x(\alpha))} \]

provided

(i) \( \alpha \) does not contain a non-base symbol
(ii) Eliminate \( \not\approx \) is not applicable to \( \alpha \)
(iii) \( x \) is a free variable in \( \alpha \)

Introduce \( \exists \) \[ N, \alpha(a) \xrightarrow{N, \exists(x(\alpha(a/x))} \]

provided

(i) \( \alpha \) does not contain a non-base symbol and it does not contain any free variables
(ii) \( a \) is a constant and does not occur in \( N \)
(iii) \( x \) is a fresh variable

Redundancy elimination \[ N, \alpha \xrightarrow{N} \]

provided

(i) there is a finite subset \( \{\beta_1, \ldots, \beta_l\} \) of \( N \) s.t. \( \models \forall x(\beta_1) \land \ldots \land \forall x(\beta_l) \) implies \( \models \forall x(\alpha) \)
(ii) \( \beta_1 \land \ldots \land \beta_l \) ‘is simpler’ than \( \alpha \)

Logical simplification \[ N, \alpha \xrightarrow{N, \beta} \]

\[ N, \gamma, \alpha \xrightarrow{N, \gamma, \beta} \]

provided

(i) \( \forall x(\alpha) \equiv \forall x(\beta) \), for the left rule
\( \forall x(\gamma \land \alpha) \equiv \forall x(\gamma \land \beta) \), for the right rule
(ii) \( \beta \) ‘is simpler’ than \( \alpha \)