Mathematical Techniques for Computer Science

Date:  Thursday 24th January 2013
Time:  09:45 - 11:15

Please answer any THREE Questions from the FOUR Questions provided.

Use a SEPARATE answerbook for each SECTION.

This is a CLOSED book examination
The use of electronic calculators is permitted provided they are not programmable and do not store text

[PTO]
SECTION A

Question 1. (Total 20 marks)

a) Consider the propositional logic formula A

\[(p \implies (q \land r)) \land ((p \land q) \implies r)\]

i) Check if A is satisfiable using the DNF test. (2 marks)
ii) If so, check if A is a tautology using the CNF test. (1 mark)
iii) For what truth valuations is A satisfiable? (2 marks)

b) Let us define the predicates

\[CS(x) \quad \text{x is a computer science student}\]
\[M(x) \quad \text{x uses Moodle}\]

Consider the statement

All Computer Science students use Moodle

i) Write this statement using predicates and quantifiers. (1 mark)

What do the following statements mean?

ii) \[\exists x \quad CS(x) \land M(x)\] (1 mark)
iii) \[\forall x \quad M(x) \implies CS(x)\] (1 mark)
iv) \[\exists x \quad CS(x) \land \neg M(x)\] (1 mark)
v) \[\forall x \quad CS(x) \implies \neg M(x)\] (1 mark)
vi) Show that the expression in (iv) is the negation of the expression in (i) (2 marks)
vii) Show that the expression in (v) is the negation of the expression in (ii) (2 marks)

c) Let us define a new connective \(\otimes\) whose truth scheme is

\[A \otimes B\text{ is true iff } A \text{ and } B \text{ have the same truth valuation}\]

(In logic, this connective is known as the XNOR connective)

i) What is the truth table of the \(\otimes\) connective? (1 marks)
ii) By writing the DNF from the truth table, express \(\otimes\) in terms of \(\neg, \land, \lor\) to verify the functional completeness of \(\neg, \land, \lor\). (1 mark)
iii) Is the \(\otimes\) connective commutative? (2 marks)
iv) Is it associative? (2 marks)

(Hint: You would be better off trying this using the truth table)
Question 2.  

(Total 20 marks)

a) Explain what is meant by a *formal language*, and name two such languages.  

(4 marks)

b) A small formal language \( R \) of *expressions* is defined by the set of rules:

The lower case letters \( a, b, \ldots, z \) are *expressions*

If \( A \) is an expression, \( \#A \) is an expression

If \( A \) and \( B \) are expressions, then so are \( \&AB, \%AB \) and \( @AB \)

i) Give three examples of expressions in this language, giving the parse tree for each.  

(3 marks)

ii) Show that the following is an expression in \( R \) by giving its parse tree:

\[ \%\&@#$\times\%ab@#\text{wvp} \]  

(6 marks)

c) Let \( a, b, c \) be fixed (but unknown) rational numbers. The binary operation \( \odot \) is defined on the set \( \mathbb{Q} \) of rational numbers by

\[ p \odot q = ap + bq + c \]

i) Give all possible combinations of values of \( a, b \) and \( c \) if we know that the operation \( \odot \) is commutative.  

(3 marks)

ii) Give all possible combinations of values of \( a, b \) and \( c \) if we know that the operation \( \odot \) is associative.  

(3 marks)

iii) Give all possible combinations of values of \( a, b \) and \( c \) if we know that the operation \( \odot \) is both commutative and associative.  

(1 marks)
 SECTION B

Question 3. (Total 20 marks)

a) A traditional fair die is rolled twice. Consider the following events

\[ A = \{ \text{a six turns up exactly once} \} \]
\[ B = \{ \text{the sum of the scores is 7} \} \]

i) Find the probabilities of the events \( \Pr(A) \), \( \Pr(B) \), \( \Pr(A \cup B) \). (6 marks)

ii) Find the conditional probability \( \Pr(A \mid B) \). (2 marks)

iii) Are the events \( A \) and \( B \) independent? Give reasons. (2 marks)

b) Every day a father gives his son 10, 20, 50 or 100 pence. Let \( X \) be the amount he gives on a particular day. We assume that \( X \) is a random variable with the probability distribution given in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( p(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.10</td>
</tr>
<tr>
<td>20</td>
<td>0.20</td>
</tr>
<tr>
<td>50</td>
<td>0.50</td>
</tr>
<tr>
<td>100</td>
<td>( c )</td>
</tr>
</tbody>
</table>

where \( c \) is some number.

i) Find the value \( c \) for which \( p(x) \) is a valid probability mass function (p.m.f.). (3 marks)

ii) Sketch the cumulative distribution function (c.d.f.) of \( p(x) \). (3 marks)

iii) What is the probability that I) \( \Pr(X < 20) \) II) \( \Pr(X > 30) \). (2 marks)

iv) Find the mean (expected value) \( E(X) \). (2 marks)
Question 4. (Total 20 marks)

a) Suppose that an archer misses a target with probability 0.1 and hits it with probability 0.9. Suppose that the archer shoots 8 times independently. Let X be the number of times the archer hits the target.

i) Find the mean and variance of X. (2 marks)

ii) Find the probability that the archer hits the target exactly five times. (2 marks)

iii) Find the probability that the archer hits the target at least seven times. (2 marks)

iv) Suppose that every time the archer hits the target he/she receives 9 pounds. Find the mean and variance of the amount of money the archer receives. (4 marks)

b) Suppose that the number of claims an insurance company receives in one day follows Poisson distribution with mean 4. Let X be the number of claims the company receives on a particular day.

i) Find the mean $E(X)$ and variance $Var(X)$. (2 marks)

ii) What is the probability that the company receives no claims on that day. (2 marks)

iii) What is the probability that the company receives at least 3 claims on that day. (3 marks)

iv) Suppose the insurance company pays off 50 pounds per claim. Find the mean and the variance of the amount the company pays off in a day. (3 marks)