Two hours

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Mathematical Techniques for Computer Science

Date: Monday 12th January 2015
Time: 09:45 - 11:45

Please answer any THREE Questions from the FOUR Questions provided.

Use a SEPARATE answerbook for each SECTION.

This is a CLOSED book examination
The use of electronic calculators is NOT permitted

[PTO]
Section A

1. a) Consider the following function $f : \mathbb{C} \rightarrow \mathbb{C}$:

   \[ x \mapsto x(2 + i) + (1 + 3i) \]

   Is this function injective, surjective, bijective? Justify your answers. (6 marks)

b)  
   i) What does it mean for a binary operation to be commutative? (2 marks)
   
   ii) Give an example of a binary operation which is not commutative. (2 marks)

c) Consider the following binary operation for complex numbers:

   \[ z \odot z' = \overline{zz'} \]

   where as usual $\overline{z}$ is the conjugate of $z$.

   i) Is this operation commutative? Justify your answer. (2 marks)
   
   ii) Is this operation associative? Justify your answer. (3 marks)

d)  
   i) What does it mean for a set to be countable? (2 marks)
   
   ii) Pick a countably infinite set which is not a subset of $\mathbb{N}$ and provide an argument that it is countable. (3 marks)
2. a) Consider the following proposition.

\((Z \rightarrow Z') \rightarrow (\neg Z \land Z' \land Z'')\).

i) Calculate a conjunctive normal form for this proposition, justifying your steps. (4 marks)

ii) Simplify your answer as far as you can. (2 marks)

iii) Is this proposition a tautology? Justify your answer. (2 marks)

iv) Is this proposition satisfiable? Justify your answer. (2 marks)

b) Consider the following system.

• variables \( x, y \) and \( z \).
• There are parameters \( 0, 1, 2, 3, 4, 5, 6 \).
• The binary function \( m \).
• A binary predicate \( E \).
• A binary predicate \( D \).

Assume that the following interpretation is applied to the system:

• The domain of interpretation is the set of natural numbers \( \mathbb{N} \),
• the parameters are interpreted by the corresponding numbers (that is, for example, \( 0 \) is interpreted by the number \( 0 \)),
• the interpretation of \( m(x, y) \) is the product \( xy \) of \( x \) and \( y \),
• the interpretation of \( E(x, y) \) is 1 if \( x = y \) and 0 otherwise and
• the interpretation of \( D(x, y) \) is 1 if \( x \) divides \( y \) and 0 otherwise.

Translate from English into predicate logic using the above system.

i) Every number is divisible by 1. (1 mark)

ii) Every number which is divisible by 4 is divisible by 2. (1 mark)

iii) Every number which is divisible by both 2 and 3 is divisible by 6. (1 mark)

iv) A number \( m \) divides a number \( n \) if and only if there exists a number \( k \) with \( km = n \). (1 mark)

For the following propositions state whether their interpretation is 1 in the given model. Justify your answers.

v) \( \forall x. \exists y. D(y, x) \). (2 marks)

vi) \( \forall x. \forall y. E(m(x, y), m(y, x)) \). (2 marks)

vii) \( \forall x. \exists y. m(y, 2) = x \). (2 marks)
3. a) Given two events $A$ and $B$, define the conditional probabilities $P(B \mid A)$ and $P(A \mid B)$. 

(5 marks)

b) The following facts are known regarding two events $A$ and $B$:

\[ P(A \cup B) = \frac{2}{3}, \quad P(A \cap B) = \frac{1}{3}, \quad P(A \mid B) = \frac{1}{2}. \]

Find the following:

i) $P(B)$; 
ii) $P(A)$; 
iii) $P(B \mid A)$; 
iv) $P(A \cap B^c)$; 
v) $P(A^c \cap B)$; 
vi) $P(A^c \cap B^c)$; 
vii) $P(A \cup B^c)$; 
viii) $P(A^c \cup B)$; 
ix) $P(A^c \cup B^c)$; 
x) $P(B \mid A^c)$; 
xi) $P(B^c \mid A)$; 
xii) $P(B^c \mid A^c)$.
4. a) State one condition that will ensure independence of two events $A$ and $B$. (3 marks)

b) If $A$ and $B$ are two independent events show the following:

i) $A$ and $B^c$ are also independent events. (3 marks)

ii) $A^c$ and $B$ are also independent events. (3 marks)

iii) $A^c$ and $B^c$ are also independent events. (3 marks)

c) A town has two fire engines operating independently. The probability that a specific fire engine is available when needed is 0.95. Find the following probabilities:

i) Probability that neither is available when needed. (3 marks)

ii) Probability that both are available when needed. (3 marks)

iii) Probability that a fire engine is available when needed. (2 marks)