Two hours

Note that the last two pages contain inference rules for natural deduction

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Mathematical Techniques for Computer Science

Date: Friday 19th January 2018

Time: 14:00 - 16:00

Please answer all THREE Questions.

Use a SEPARATE answerbook for each SECTION.

This is a CLOSED book examination

The use of electronic calculators is permitted provided they are not programmable and do not store text

Section A

1. a) Prove or disprove the following statement: For i, j and k in $\mathbb{Z}$ we have that (2 marks)

$$i \text{ divides } jk \qquad \text{implies} \qquad i \text{ divides } j \text{ or } i \text{ divides } k.$$

b) Show that for all $r \in \mathbb{R}$ and $z \in \mathbb{C}$ we have that (2 marks)

$$|rz| = |r||z|.$$

c) Consider the following binary operation on the set of integers $\mathbb{Z}$:

$$m \circledast n = (m \bmod 2) \cdot n.$$

i) Is the given operation commutative? Give a reason for your answer. (2 marks)

ii) Is this operation associative? Justify your answer. (4 marks)

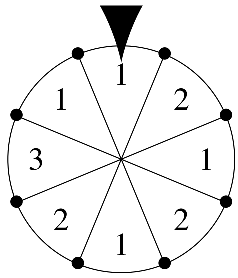
d) Consider the following function.

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \begin{cases} \frac{1}{|z|^2}z & z \neq 0 \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Is this function injective? Is it surjective? Justify your answers. Give an informal argument if you cannot give a formal one. (8 marks)

e) You have learned what it means for one function from $\mathbb{N}$ to $\mathbb{N}$ is eventually dominated by another. Under which circumstances might a computer scientist be interested in that idea? (2 marks)

2. Assume you have a random device in the form of a wheel as pictured here.



The wheel rotates around its centre point and when it stops we read off the number that the arrow points at. There are dividers that ensure that the arrow cannot point at the borderline between two fields. You should assume that what we mean by 'the wheel is turned' is a random process for which each sector has an equal chance to be the one at which the arrow points.

The idea of a wheel is new to the students, but they have practised calculations similar to the ones below.

a) Give a probability space that describes the wheel being turned. (2 marks)

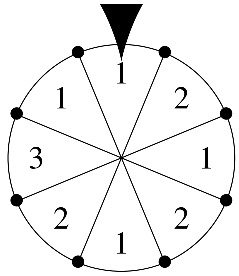
b) You are being offered a game that consists of turning the wheel twice and adding up the two numbers. Should you bet on the result being even or odd? (2 marks)

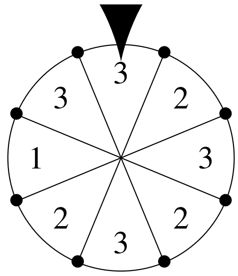
c) You are being offered a similar game, only this time the two numbers are multiplied. Should you bet on the result being even or odd? (2 marks)

d) Consider the random variable X that is the result of turning the wheel twice and adding up the two resulting numbers. What is the expected value of that number? (1 mark)

e) What is the expected value of the sum of the numbers shown by turning the wheel twice and adding up the two resulting numbers, given that the first number is odd? (3 marks)

f) One of your friends has built a wheel, which is either like the one pictured above, or it is one where the values 3 and 1 are exchanged, so it is one of these.





You want to carry out Bayesian updating to work out which kind of wheel he has. He will turn the wheel for you and tell you the outcome, without showing you the wheel itself. Carry out Bayesian updating step by step, assuming that your friend turns the wheel and gets 3, and turns it again and gets 2, and turns it again and gets 3. (10 marks)

Section B

3. a) Construct a truth table for the formula $Q \to (P \leftrightarrow \top)$. (2 marks)

b) Give a brief explanation of **two** of the following. (4 marks)

i) truth table
ii) disjunctive normal form
iii) Boolean function

c) Consider the following propositional formula.

$$(P \vee Q) \to (\neg(P \to Q) \vee R)$$

i) Give a conjunctive normal form for the formula. (3 marks)
ii) Simplify the conjunctive normal form as far as possible. (3 marks)

Explain all the steps in your calculations.

d) Give a natural deduction proof for the following. (3 marks)

$$\vdash (P \wedge \neg Q) \to (P \vee Q)$$

Use the inference rules of our natural deduction system, which are given on the last pages of this exam paper.

e) Let F be the following first-order formula.

$$\forall x\big((L(x) \wedge G(x, j)) \to \forall z(S(z) \to A(z,x))\big)$$

Suppose the non-logical symbols in the formula have the following interpretation.

$L(x)$	means that x is a lecture
$S(x)$	means that x is a student
$G(x,y)$	means that x gives y
$A(x,y)$	means that x attends y
j	represents Jim

i) Express the formula F in idiomatic English.
ii) Negate the formula F and simplify it as far as possible using the fundamental laws of first-order logic and propositional logic.
iii) Express the simplified form of the negation of the formula in idiomatic English.

(5 marks)

Rules of inference of our propositional natural deduction system

Conjunction elimination:

If $A \wedge B$ is derivable from a set of formulas, then so is A, and also B.

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

Conjunction introduction:

If A is derivable from a set of formulas, and B is derivable from the same set, then $A \wedge B$ is derivable from this set as well.

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

Disjunction introduction:

If A is derivable from a set, then so is $A \vee B$, and also $B \vee A$.

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash B \vee A}$$

Disjunction elimination (proof by cases):

If $A \vee B$ is derivable from a set and C is derivable from the set along with A, and also from the set along with B, then C is derivable from the set alone.

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

Implication introduction:

If B is derivable from A and a set, then $A \to B$ is derivable from the set .

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

Implication elimination:

If A is derivable from a set, and $A \to B$ is derivable from the same set, then B is derivable from this set.

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash A \to B}{\Gamma \vdash B}$$

Negation introduction (reductio ad absurdum):

If A and a set leads to a contradiction, then $\neg A$ can be inferred from the set.

$$\frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}$$

Negation elimination:

If A is derivable from a set, and also $\neg A$ is derivable from the set, then anything (including $\bot$) is derivable from the set.

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash \neg A}{\Gamma \vdash B}$$

Double negation introduction:

If A is derivable from a set, then $\neg\neg A$ is derivable from the same set.

$$\frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A}$$

Double negation elimination:

If $\neg\neg A$ is derivable from a set, then A is derivable from the same set.

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

Axiom (starting point):

A can always be inferred from A and a set of formulas.

$$\frac{}{\Gamma, A \vdash A}$$

Weakening:

New assumptions may be introduced at any point in a derivation.

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$

END OF EXAMINATION