Two hours

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Fundamentals of Computation

Date: Tuesday 27th May 2014
Time: 14:00 - 16:00

Please answer any THREE Questions from the FOUR Questions provided.

Use a SEPARATE answerbook for each SECTION.

This is a CLOSED book examination

The use of electronic calculators is NOT permitted

[PTO]
1. a) Consider the language of all words over the alphabet \( \{a, b\} \) which have length at least 2 and whose first letter is equal to its penultimate (that is last but one) letter. Give a description of this language via the following means.

i) using a regular expression (2 marks)
ii) using a DFA (4 marks)
iii) using a grammar (3 marks)
iv) using the language of set theory (2 marks)

b) Consider the following DFA over the alphabet \( \{a, b, c\} \).

![DFA Diagram]

i) Give a regular expression that defines the same language. (7 marks)
ii) Shorten your regular expression from (ii). (2 marks)
2. a) Consider the following grammar. The underlying alphabet is \{a, b, c\}, there are three non-terminal symbols, \(S, T \) and \(U\), and the production rules are:

\[
S \rightarrow aT \\
T \rightarrow UaU \\
U \rightarrow aU \mid bU \mid cU \mid \epsilon
\]

i) Is this grammar ambiguous? Justify your answer. (3 marks)

ii) Give a non-trivial property shared by all the words in the language generated by the grammar. (2 marks)

iii) Describe the language generated by the grammar. (3 marks)

iv) Give an unambiguous grammar for this language. (3 marks)

b) For the following NFA give a DFA that accepts the same language. (3 marks)

c) For the following automata, give two simulations, one going from the one on the left to the one on the right and one going in the opposite direction, or argue that in the given direction it is impossible to do so. (6 marks)
Section B

3. a) State the principle of structural induction for finite lists (2 marks)

b) Using structural induction, prove that

\[
\text{reverse } (xs ++ ys) = \text{reverse } ys ++ \text{reverse } xs
\]

Append (++) and reverse are defined as:

\[
(+ +) :: [a] \rightarrow [a] \rightarrow [a]
\]

\[
[] ++ ys = ys \\
(x:xs) ++ ys = x: (xs ++ ys)
\]

\[
\text{reverse} :: [a] \rightarrow [a]
\]

\[
\text{reverse } [] = [] \\
\text{reverse } (x:xs) = \text{reverse } xs ++ [x]
\]

You may assume that ++ is associative. (5 marks)

c) Use structural induction to prove that

\[
\text{rev } xs = \text{reverse } xs
\]

Assume that \text{rev} is defined as follows.

\[
\text{rev} :: [a] \rightarrow [a] \\
\text{rev } xs = \text{shunt } [] \text{ } xs
\]

where

\[
\text{shunt} :: [a] \rightarrow [a] \rightarrow [a] \\
\text{shunt } ys [] = ys \\
\text{shunt } ys (x:xs) = \text{shunt } (x:ys) \text{ } xs
\]

The new implementation of \text{rev} uses a new recursive function \text{shunt} to move elements from the second list into the first list in the reverse order.

Why do you need to use an auxiliary lemma? (5 marks)

d) Show that there is a bijection between the set of arithmetic expressions in the \text{while} programming language and the set of natural numbers \mathbb{N}. You may assume that there is a bijection \(\phi_v\) between the variables of \text{while} and the natural numbers \mathbb{N}.

Explain why your answer is a bijection; you need not formally prove this (5 marks)

e) Explain why this shows that the set of arithmetic expressions is \text{countable}. (3 marks)
4. a) Explain the following concepts: pre-condition, post-condition, partial correctness, and loop-invariant with reference to the `while` language. (2 marks)

b) Consider the following `while` program:

```
(while 1 <= x do x := x-2); r := x
```

Show that the remainder of $x$ on division by two is a loop-invariant for the while loop. (5 marks)

c) What is the value of $r$ when the program from the previous part terminates? Using the loop invariant, prove your hypothesis. (3 marks)

d) Explain the differences between $O$-, $\Omega$-, and $\Theta$-notations for algorithmic efficiency. (2 marks)

e) State which of the following are true, and provide reasoning to support your conjectures.

```
\begin{align*}
x & \in O(x^2) \\
x + x^2 & \in \Theta(x^2) \\
x & \in \Omega(x^2) \\
x & \in O(x + 7) \\
2 & \in O(1)
\end{align*}
```

(5 marks)

f) Show that if $f \in O(g)$ and $f \in \Omega(g)$ then $f \in \Theta(g)$. (3 marks)