Two hours

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Fundamentals of Computation

Date: Thursday 26th May 2016

Time: 09:45 - 11:45

Please answer any THREE Questions from the FOUR Questions provided.

Use a SEPARATE answerbook for each SECTION.

This is a CLOSED book examination

The use of electronic calculators is NOT permitted

Section A

1. a) Consider the language $\mathcal{L}$ of all words over the alphabet $\{a,b,c\}$ in which every occurrence of the letter b is both preceded *and* followed by the letter a.

   Give a description of $\mathcal{L}$ via the following means:

   i) using a DFA (3 marks)
   ii) using a grammar (2 marks)
   iii) using a regular expression (2 marks)
   iv) using the language of set theory (2 marks)

   b) Give a DFA for the language of all words over the alphabet $\{a,b,c\}$ that match the regular expression

$$a^*bc^* \mid ab^*c$$

(8 marks)

   c) For a language $\mathcal{L}$ over an alphabet Σ, we can construct the language $\mathcal{L}^R$:

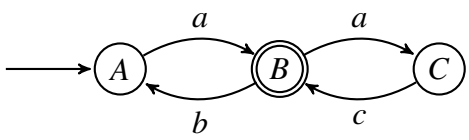
$$\mathcal{L}^R = \{x_n x_{n-1} \cdots x_2 x_1 \mid x_1 x_2 \cdots x_{n-1} x_n \in \mathcal{L}\}.$$

   Proposition: For every regular language $\mathcal{L}_1$, the language $\mathcal{L}_1^R$ is also regular.

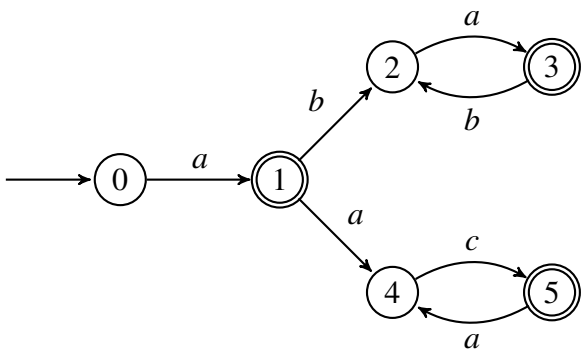
   Sketch out a proof for the above proposition. (3 marks)

2. a) Consider the two DFAs X and Y given below.

X



Y



Proposition I These two automata are **not** equivalent.

Proposition II By adding one or more transitions to automaton Y, we can produce a new DFA Y' which **is** equivalent to X.

i) Provide a proof for *Proposition I* above, i.e. demonstrate that the automata are not equivalent. (2 marks)

ii) Draw an automaton that satisfies the conditions for Y'. (4 marks)

iii) Demonstrate that *Proposition II* holds, i.e. that X and your proposed automaton Y' are equivalent. (4 marks)

b) Are the following languages over the alphabet $\{a,b\}$ regular? Justify your answer.

i) The language $\mathcal{L}_1 = \{a^n \mid n \in \mathbb{N},\ n \text{ is even}\}$. (2 marks)

ii) The language $\mathcal{L}_2 = \{a^n b^n \mid n \in \mathbb{N},\ n \text{ is even}\}$. (2 marks)

c) Consider the following grammar. The underlying alphabet is $\{a,b,c\}$, there are two non-terminal symbols S and T, the start symbol is S and the production rules are:

$$\begin{array}{rcl} S & \rightarrow & TabT \\ T & \rightarrow & aT \mid bT \mid cT \mid \varepsilon \end{array}$$

i) Show that this grammar is ambiguous. (2 marks)

ii) Describe the language generated by this grammar (1 mark)

iii) Give an unambiguous grammar for the same language (3 marks)

Section B

3. a) The following program calculates a function $f(\mathtt{n})$ for each input value `n`, with the result being stored in variable `a`. Tabulate the function for values of `n` satisfying $0 \leq \mathtt{n} \leq 5$.

```
if n = 0 then a := 0 else a := 1;
b := 0;
while 2 <= n do
  ( tmp := a;
    a := a + b;
    b := tmp;
    n := n - 1
  )
```

(4 marks)

b) Using induction or otherwise show that the function implemented in part (a) can be defined recursively as:

$$\begin{aligned} f(0) &= 0, \\ f(1) &= 1, \\ f(n+2) &= f(n+1) + f(n). \end{aligned}$$

(You need not use pre- and post-conditions, but you should explain the relationship of your proof to the program.) (6 marks)

c) Let

$$\begin{aligned} \phi &= \tfrac{1}{2}(1+\sqrt{5}), \\ \hat{\phi} &= \tfrac{1}{2}(1-\sqrt{5}) = 1-\phi. \end{aligned}$$

Using induction or otherwise show that

$$f(n) = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right).$$

(6 marks)

d) Using the fact that $|\hat{\phi}| < 1$, argue that the function f is in $O(\phi^n)$.

(4 marks)

4. a) Explain the following concepts: computable function; decidable predicate; semi-decidable predicate, and the characteristic function of a predicate. (4 marks)

   b) What is the Church-Turing Thesis? Argue that testing whether a number is a prime number is a decidable function. (6 marks)

   c) What is the Diagonal Program? (6 marks)

   d) i) Could you write a program which takes as input the source code of a `while` program and a value and outputs whether or not that program will halt when run with that value as input? If not, why not? (1 mark)

      ii) If there was a special `halt` construct added to the `while` language that told us whether or not a program would halt when run on a particular input value, what can you say about the resulting programming language? (3 marks)

END OF EXAMINATION