Two hours

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Mathematical Techniques for Computer Science

Date: Monday 5th June 2017

Time: 09:45 - 11:45

Please answer all THREE Questions.

Use a SEPARATE answerbook for each SECTION.

This is a CLOSED book examination

The use of electronic calculators is permitted provided they are not programmable and do not store text

Section A

1. a) Consider the function $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ recursively determined by

Base case f. $\quad f0 = 5.$

Step case f. $\quad f(n+1) = fn + 2n + 1.$

i) Calculate the first five values of f. (1 mark)

ii) Give a non-recursive description of the function f. (2 marks)

iii) Prove by induction that your answer from part ii) satisfies the original recursive specification for all $n \in \mathbb{N}$. (4 marks)

b) Consider the set of lists over $\mathbb{Z}$, Lists_Z, from the notes.

i) Recursively define a function

$$\mathrm{ins}_1 : \mathsf{Lists}_Z \longrightarrow \mathsf{Lists}_Z,$$

which inserts a 1 before each of the entries of the given list, so that we have, for example, (3 marks)

$$\mathrm{ins}_1 [3,2,1,0] = [1,3,1,2,1,1,1,0].$$

ii) In the notes there is a Java class List which is the direct counterpart of a recursively defined list. For this class give a method that implements the function described in part i) by filling in the two return cases in the following Java program. (2 marks)

```
public static List ins1 (List l)
{
if (l == null)
    return
else
    return
}
```

iii) Recall the operator $\mathrm{sum}\colon \mathsf{Lists}_Z \longrightarrow Z$ given by

Base case sum. $\mathrm{sum}\,[\,] = 0.$

Step case sum. $\mathrm{sum}(s:\ l) = s + \mathrm{sum}\, l$

and the operator $\mathrm{len}\colon \mathsf{Lists}_Z \longrightarrow \mathbb{N}$ given by

Base case len. $\mathrm{len}\,[\,] = 0$

Step case len. $\mathrm{len}(s:\ l) = 1 + \mathrm{len}\, l.$

Show that for all $l \in \mathsf{Lists}_Z$ (4 marks)

$$\mathrm{sum}\,\mathrm{ins}_1\, l = \mathrm{sum}\, l + \mathrm{len}\, l.$$

c) Show by induction that for all $n \in \mathbb{N}$ we have that (4 marks)

$$n^5 = n \pmod 5.$$

2. a) Consider the following binary relations for binary strings. (6 marks)

   i) Check whether the given relations are equivalence relations. Justify your answer.

   - Firstly, the relation

   $$l \sim l' \qquad \text{iff} \qquad l \text{ and } l' \text{ encode the same number in base 2.}$$

   - Secondly, the relation

   $$l \sim l'$$

     if and only if
     – the first symbol of l is different from the first symbol of l' and
     – the last symbol of l is different from the last symbol of l'.

   ii) For any equivalence relations identified in i), describe the equivalence class of 1.

b) Consider the following relation on the set of binary strings.

$$l \leq l'$$

if and only if the number of 1s in l is less than or equal to the number of $1s$ in l'.

Is this a partial order? If your answer is yes then draw a Hasse diagram for the strings 000, 011, 100 and 101. (3 marks)

c) Assume you are a member of a team that has to write a program that has to be able to perform Tasks A to F. After studying the problem you and your team have worked out the following:

- The output of Task A is required for Tasks B and D.
- Task E relies on the output of Tasks C and D.

Draw a Hasse diagram for the reflexive closure of the 'output of Task X is required for Task Y' relation, which is a partial order. Identify the minimal and least elements.

What can you say about the minimal elements of the partial order when it comes to planning the design, implementation and testing of your code? (4 marks)

d) Recall the set $\mathsf{FBTrees}_{\mathbb{N}}$ of full binary trees with labels from $\mathbb{N}$. We recursively define an equivalence relation on $\mathsf{FBTrees}_{\mathbb{N}}$ as follows.

Base case tree. For all $n \in \mathbb{N}$, $\mathtt{tree}\, n \sim \mathtt{tree}\, n$.

Step case tree. If $n \in \mathbb{N}$, and if we have

$$t \sim t'' \qquad \text{and} \qquad t' \sim t'''$$

then

$$\mathtt{tree}_n(t,t') \sim \mathtt{tree}_n(t'',t''') \qquad \text{and} \qquad \mathtt{tree}_n(t,t') \sim \mathtt{tree}_n(t''',t'').$$

i) Draw the following trees.

$\mathtt{tree}_3(\mathtt{tree}\,4, \mathtt{tree}\,5)$, $\quad$ $\mathtt{tree}_3(\mathtt{tree}\,5, \mathtt{tree}\,4)$,
$\mathtt{tree}_3(\mathtt{tree}_4(\mathtt{tree}\,2, \mathtt{tree}\,1), \mathtt{tree}\,5)$, $\quad$ $\mathtt{tree}_3(\mathtt{tree}_5(\mathtt{tree}\,2, \mathtt{tree}\,1), \mathtt{tree}\,4)$,
$\mathtt{tree}_3(\mathtt{tree}\,5, \mathtt{tree}_4(\mathtt{tree}\,1, \mathtt{tree}\,2))$

Which of them are $\sim$-equivalent? (3 marks)

ii) Describe what it means for two trees to be $\sim$-equivalent without using recursion. (4 marks)

Section B

3. a) Let $\underline{a}$ and $\underline{b}$ be two vectors emanating from one point X and suppose A and B are the endpoints of $\underline{a}$ and $\underline{b}$ respectively. Let C be the point reached by extending the vector emanating from point A and with the endpoint being B by a factor of n.

   i) Express the vector emanating from X and with the endpoint at C in terms of $\underline{a}$ and $\underline{b}$. (2 marks)

   ii) Compute the coordinates of C if the coordinates of A are $(1,1)$ and the coordinates of B are $(3,2)$. (1 mark)

   b) Consider the following matrices.

$$\underline{A} = \begin{pmatrix} -1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \qquad \underline{B} = \begin{pmatrix} \frac{1}{2} & \pi \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   Compute the following, if it is defined. (3 marks)

   i) $2\underline{A}2\underline{B}$

   ii) $4\underline{AB} - \underline{I}_2$

   c) Give the geometric interpretation of the linear transformation T represented by the following matrix. (2 marks)

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   d) For each of the following statements state whether it is true or false. Give an explanation if the statement is true. Give a counter-example if the statement is false. (8 marks)

   i) If, when multiplying two matrices we obtain a vector, then the second matrix is a vector. (By a vector we mean a standard column vector.)

   ii) It is not the case that $\underline{AB} = \underline{BA}$ for any matrices $\underline{A}$ and $\underline{B}$.

   iii) To specify a linear transformation it is sufficient to give where the standard basis vectors are mapped.

   iv) If a homogeneous system $\underline{Ax} = \underline{0}$ of linear equations has $\underline{0}$ as its only solution then $\underline{Ax} = \underline{b}$ has a unique solution for each vector $\underline{b}$ with the same number of entries as $\underline{A}$ has rows.

e) Suppose

$$\underline{A} = \begin{pmatrix} 1 & 0 & -3 & 2 \\ 2 & -1 & -6 & 8 \\ -1 & 2 & 3 & 6 \end{pmatrix}$$

is the augmented matrix of a system of linear equations.

i) Write down the system of linear equations represented by $\underline{A}$. (1 mark)

ii) Use the Gaussian elimination method to solve the system and find all solutions. (3 marks)

END OF EXAMINATION