Quantum Computing

Date: Friday 25th May 2018
Time: 09:45 - 11:45

Please answer all Questions.

This is a CLOSED book examination

The use of electronic calculators is NOT permitted

[PTO]
1. a) What is a basis? (2 marks)

b) What is an orthonormal basis? (2 marks)

c) Prove the following Triangle Inequality: \(|x| - |y| \leq |x+y|\). (4 marks)

d) How are the vectors in \(\mathbb{C}^n\) characterised? (1 mark)

e) How are the operators in \(\mathbb{C}^n\) characterised? (1 mark)

f) How are the coefficients \(\lambda_k\) calculated in the expansion \(|x\rangle = \sum_k \lambda_k |k\rangle\) of an arbitrary vector \(|x\rangle\) when the \(|k\rangle\) form an orthonormal basis? (2 marks)

g) Show that if \(n\) is a 3D unit vector, then \((n.\sigma)^2 = I\). (3 marks)

h) Show that if \(n\) is a 3D unit vector, then \(\exp(i\theta n.\sigma) = \cos(\theta)I + i\sin(\theta)n.\sigma\). (5 marks)
2. a) Write down the axioms of quantum mechanics that define: (i) the state space of a system, (ii) the state space of a composite system versus its components, (iii) the evolution of an isolated system over a period of time. (4 marks)

b) An arbitrary qubit state can be written as \([\cos(\theta/2), e^{i\phi} \sin(\theta/2)]^T\). Derive this. (4 marks)

c) For 1-qubit systems, the operator \(n \cdot \sigma = (n_x \sigma_x + n_y \sigma_y + n_z \sigma_z)\) is very significant. Explain why. (2 marks)

d) Write down the matrix representation of the operator \(n \cdot \sigma\). (2 marks)

e) Let \(I\) and \(X\) be the identity and NOT operators on a 1-qubit state space. What are the eigenvalues and eigenvectors of \(I \oplus X\) ? What are the eigenvalues and eigenvectors of \(I \otimes X\)? (2 marks)

f) Assume local realistic physics (and, should you need it, counterfactual reasoning). Let \(A, B, C\), be outcomes of classical experiments that can produce results +1 or −1 only. By examining the possible cases, deduce that if ensemble averages are calculated for a large set of trials of these experiments, the inequality \(|\langle BA \rangle + \langle BC \rangle| \leq 1 + \langle AC \rangle\) is satisfied. (3 marks)

g) Following on from part (f), now reinterpret the ensemble averages as quantum mechanical expectations (and use as much counterfactual reasoning as you might need). Using the formula \(\langle PQ \rangle = -\cos(\angle(P, Q))\), show that the inequality of part (f) can be violated. (3 marks)
3.  
   a) Write down the matrices for the I, X, Y, Z, Φ₀ operators. (4 marks)
   
   b) How do the four standard basis vectors get mapped by the CNOT gate? (2 marks)
   
   c) Describe what happens to an arbitrary vector in \( \mathbb{Q} \otimes \mathbb{Q} \) when a measurement is made on the first qubit. (2 marks)
   
   d) Let \( V = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \).
   
   What does the following circuit do? Justify your answer. (4 marks)

   \[ \begin{array}{ccc}
   \text{H} & \text{V} & \text{V} & \text{H} \\
   \end{array} \]

   e) In the quantum order finding algorithm, write down the unitary transformation \( U \) that plays the key role. (4 marks)
   
   f) Referring to part (e), many of the eigenvectors of \( U \) depend on the order \( r \), which is the goal. Why is progress possible despite this? (2 marks)
   
   g) What is the elementary algebraic identity that links knowledge of the order \( r \) of \( x \) modulo \( N \) to the factorisation of \( N \)? (2 marks)

END OF EXAMINATION