COMP 11120

Two hours

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Mathematical Techniques for Computer Science

Date: Thursday 23rd May 2019

Time: 14:00 - 16:00

Please answer all THREE Questions
Each question is worth 20 marks

Use a SEPARATE answer book for each SECTION

This is a CLOSED book examination

The use of electronic calculators is permitted provided they are not programmable and do not store text

[PTO]

Section A

1. a) Show by induction that for all $n \in \mathbb{N}$ we have (4 marks)

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

b) Consider the set $\mathsf{Lists}_{\mathbb{N}}$ of lists with entries from $\mathbb{N}$ from the notes.

i) For a natural number $b \in \mathbb{N}$, recursively define a function

$$\mathrm{occ}_b \colon \mathsf{Lists}_{\mathbb{N}} \longrightarrow \{0,1\},$$

which returns 1 if b occurs in the input list and 0 otherwise. (3 marks)
For example, with $b = 2$ we have

$$\mathrm{occ}_2[1,4,2,4] = 1,$$
$$\mathrm{occ}_2[1,3,5,7] = 0.$$

ii) In the notes there is a Java class List (with instance variables called value and next) which is the direct counterpart of a recursively defined list. For this class give a method that implements the function described in part i) by filling in the return cases in the following Java program, where the number b is passed as the second argument. (3 marks)

```
public static int occ (List l, int b)
{
if (l == null)
    return
else



}
```

iii) For a natural number $a \in \mathbb{N}$ consider the following operation zer_a on lists, which replaces all occurrences of a in a list with 0:

Base case zer_a. $\mathrm{zer}_a([\,]) = [\,]$.

Step case zer_a. $\mathrm{zer}_a(s : l) = \begin{cases} 0 : \mathrm{zer}_a(l) & \text{if } s = a \\ s : \mathrm{zer}_a(l) & \text{otherwise} \end{cases}$

Show that for all $l \in \mathsf{Lists}_{\mathbb{N}}$ we have that (4 marks)

$$\mathrm{occ}_3(\mathrm{zer}_3(l)) = 0$$

c) What does it mean for a set to be infinite? Consider the finite set $A = \{1,2,3\}$. Is the set Lists_A finite or infinite? Give a proof that your answer is correct.
(6 marks)

Section B

2. a) Check whether the relations defined below are equivalence relations on the set of non-empty binary strings. Justify your answers. (6 marks)

   i) String s is related to string t if and only if
   - the number of 0s in s is equal to the number of 0s in t and
   - the number of 1s in s is equal to the number of 1s in t.

   ii) String $a_0a_1\cdots a_m$ is related to string $b_0b_1\cdots b_n$ (with each $a_i, b_j \in \{0,1\}$) if and only if
   - $m = n$ and
   - for all $0 \leq i \leq m-1$ we have $a_i = b_{m-i}$.

   For any equivalence relation you find give the equivalence class of 101.

   b) On the set $\{a,b,c,d,e\}$ consider the relation given by the pairs

   $$\{(a,b), (c,d), (a,c), (a,e), (e,d)\}$$

   i) Draw the reflexive transitive closure of this relation. (1 mark)

   ii) Is the relation from the previous part a partial order? If yes draw its Hasse diagram. (2 marks)

   iii) Determine any greatest, least, minimal and maximal elements. (3 marks)

   c) On the set of all binary strings consider the partial order given recursively as follows: For $a,b \in \{0,1\}$, where $0 < 1$, and binary strings s and t

   Base cases If s is the empty string then $s \leq t$.
   Base cases If $a < b$ then $as \leq bt$.
   Step cases If $s \leq t$ then $as \leq at$.

   i) Draw a Hasse diagram for the strings 0, 1, 00, 01 and 10. (2 marks)

   ii) Identify a non-trivial property of the partial order given, and give an argument that it does indeed have that property. (3 marks)

   d) For each of the elements of the quotient set $\mathbb{Z}/_{\sim}$, where $\sim$ is equivalence modulo 7, calculate the multiplicative inverse if it exists. (3 marks)

3. a) For each of the following statements state whether it is true or false. Give an explanation if the statement is true. Give an explanation or a counter-example if the statement is false. (8 marks)

   i) $\underline{v} + (\underline{w} - \underline{v}) = \underline{w}$ for any vectors $\underline{v} \in \mathbb{R}^n$ and $\underline{w} \in \mathbb{R}^n$.

   ii) It is not possible to express $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

   iii) Let $T_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the function defined by $T_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ 0 \end{pmatrix}$, and

   let $T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the function defined by $T_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-2y \\ \pi x \\ y^2 \end{pmatrix}$.

   Both T_1 and T_2 are linear transformations.

   iv) The system of linear equations associated with this augmented matrix has infinitely many solutions.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 0 \\ 9 & 10 & 11 & 12 & 0 \end{pmatrix}$$

b) Let $\underline{A}$ be any $m \times n$ matrix and $\underline{u} \in \mathbb{R}^n$ the vector in which every entry is 1.

   i) Show that multiplying $\underline{A}$ with $\underline{u}$ gives a vector in which the ith entry is the sum over the entries in the ith row of $\underline{A}$. (2 marks)

   ii) Say how the averages of every *column* in $\underline{A}$ can be computed using matrix operations. (2 marks)

c) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ x \\ x-y \end{pmatrix}$, and

   let $T' : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by $T' \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y+z \\ x+y \end{pmatrix}$.

   i) Find the standard matrix of the linear transformation $T' \circ T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$.

   ii) Find a formula for $(T' \circ T) \begin{pmatrix} x \\ y \end{pmatrix}$.

(3 marks)

d) Consider the following system of linear equations.

$$\begin{aligned} 2x+4y-2kz &= 0 \\ -2y+kz &= -2 \\ x+2y &= 3 \end{aligned}$$

i) Determine for which values of $k \in \mathbb{R}$ does the system have solutions. (3 marks)

ii) Compute the solution set for the system. (2 marks)

END OF EXAMINATION