Please answer any THREE Questions from the FOUR questions provided.

Use a SEPARATE answerbook for each SECTION.

For full marks your answers should be concise as well as accurate.

Marks will be awarded for reasoning and method as well as being correct.

The use of electronic calculators is permitted provided they are not programmable and do not store text.
1. a) i) Prove by induction that, for all integers \(n \geq 0\)

\[
\sum_{i=0}^{n} (2i + 1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}
\]

(5 marks)

Model answer and marking scheme
Let
\[
\begin{align*}
L(n) &= \sum_{i=0}^{n} (2i + 1)^2 \\
R(n) &= \frac{(n+1)(2n+1)(2n+3)}{3}
\end{align*}
\]

Base case:-
\(L(0) = 1^2 = 1\) and \(R(0) = 3/3 = 1\) (1 mark)

Induction step:-
Suppose, for some \(n\), \(L(n) = R(n)\)

\[
L(n+1) = \sum_{i=0}^{n+1} (2i + 1)^2 \\
= \sum_{i=0}^{n} (2i + 1)^2 + (2n + 3)^2 \\
= L(n) + (2n + 3)^2 \\
= R(n) + (2n + 3)^2 \\
= \frac{(n+1)(2n+1)(2n+3)}{3} + (2n + 3)^2 \\
= \frac{2n+3}{3}((n + 1)(2n + 1) + 3(2n + 3)) \\
= \frac{2n+3}{3}(2n^2 + 3n + 1 + 6n + 9) \\
= \frac{2n+3}{3}(2n^2 + 9n + 10) \\
= \frac{2n+3}{3}(n + 2)(2n + 5) \\
= \frac{(n+2)(2n+3)(2n+5)}{3} \\
= R(n + 1)
\]

(4 marks)

Marker’s feedback
This part of the question was straightforward and generally answered well. Quite a number of students lost the odd mark for not giving enough detail in their proof. A sudden leap to the ‘right’ answer without any justification was penalised.

ii) Prove, by induction that \(4^{n+1} + 5^{2n-1}\) is divisible by 21 for all \(n \geq 1\). (5 marks)
Model answer and marking scheme
Let $f(n) = 4^{n+1} + 5^{2n-1}$.

Base case:
$f(1) = 4^2 + 5 = 21$, which is divisible by 21. (1 mark)

Induction step:

$$f(n+1) = 4^{n+2} + 5^{2n+1}$$
$$= 4^{n+1} + 25.5^{2n-1}$$
$$= 4^{n+1} + 4.5^{2n-1} + 21.5^{2n-1}$$
$$= 4f(n) + 21.5^{2n-1}$$

Since $f(n)$ is divisible by 21, so is $f(n+1)$ (4 marks)

Marker’s feedback
This caused much more trouble. The base case was easy, although some tried to use $n = 0$ for which the property isn’t true.
b) i) Determine whether each of the following functions \( \mathbb{Z} \rightarrow \mathbb{Z} \) is injective, surjective, neither or both. Justify your answers.

- \( f(n) = n - 1 \)
- \( g(n) = n^2 + 1 \)
- \( h(n) = 2n + 1 \)

(6 marks)

Model answer and marking scheme

- \( f \) is both injective and surjective. (2 marks)
- \( g \) is neither injective and surjective. (2 marks)
- \( h \) is injective but not surjective. (2 marks)

Marker’s feedback

Not badly answered although there does seem to be a fairly widespread belief that a function has to be either injective or surjective (not ‘subjective’ as one suggested) and that one is the negation of the other. This despite notes and slides giving loads of examples to show that this is not the case. Some students failed to notice that the source and target of all these functions was \( \mathbb{Z} \) and argued using graphs of functions from \( \mathbb{R} \) to \( \mathbb{R} \).

ii) If functions \( p \) and \( q \) have the properties that both \( p \) and \( p \circ q \) are surjective, does it follow that \( q \) is? Justify your answer. (4 marks)

Model answer and marking scheme

\( q \) need not be surjective. (1 mark)

See counterexample.

Marker’s feedback

Very poorly answered in general. A simple counter-example was all that was needed.
2. a) For each of the following relations, determine whether it is reflexive, whether it is symmetric and whether it is transitive. If the relation is an equivalence relation, describe its equivalence classes.

i) The relation \( R \) on the set \( X = \{a, b, c, d, e, f\} \) is given by

\[
R = \{(a, a), (b, b), (b, d), (c, c), (d, d), (d, b), (e, e), (e, f), (f, f), (f, e)\}
\]

(4 marks)

Model answer and marking scheme
The digraph of \( R \) is

- \( R \) is reflexive since, \( \forall x \in X, (x, x) \in R \)
- \( R \) is symmetric since, \( \forall x, y \in X, (x, y) \in R \Rightarrow (y, x) \in X \)
- \( R \) is transitive since, \( \forall x, y, z \in X, (x, y) \in R \land (y, z) \in R \Rightarrow (x, z) \in R \)

Equivalence classes are the connected components of the digraph. (4 marks)

Marker's feedback
The easiest way to answer this was by drawing a digraph

ii) The relation \( C \) on the set of Manchester Undergraduate Students defined by

\( x C y \) iff \( x \) and \( y \) take at least one course in common.

(4 marks)
Model answer and marking scheme

- $C$ is reflexive since every student takes at least one unit in common with themselves. This assumes that every student takes at least one unit; if we don’t have this assumption then $C$ is not reflexive. Either answer acceptable with justification. (1 mark)

- $C$ is symmetric since, if $x$ and $y$ take at least one course in common, so do $y$ and $x$. (1 mark)

- $C$ is not transitive, since, if $x$ and $y$ take at least one course in common and $y$ and $z$ take at least one course in common, it may not be the case that $x$ and $z$ take at least one course in common. The common course shared by $x$ and $y$ might not be the same as that shared by $y$ and $z$. (2 marks)

Marker’s feedback
Most answered this part quite well although some lost marks for not giving any justification for their answers.
b) A relation $T$ is defined on the set \{a, b, c, d\} by

$$T = \{(a,b), (a,c), (c,d), (c,a)\}$$

Model answer and marking scheme
Digraph of $T$ is

![Digraph of T](image)

i) Find the reflexive closure of $T$

Model answer and marking scheme
Digraph is

![Digraph is](image)

(1 mark)

ii) Find the symmetric closure of $T$

Model answer and marking scheme
Digraph is

![Digraph is](image)

(1 mark)
iii) Find the transitive closure of $T$

Model answer and marking scheme

Digraph is

(2 marks)

(4 marks)

Marker’s feedback

Many students gave an answer which suggested that the closure (of whatever sort) consists of just the extra edges that need to be added. This is not the case, the closure is the union of the original relation and the extra edges.

c) Let $S$ denote the set of bit strings of length 3 or more. Define a relation $K$ on $S$ by

$$s_1 K s_2 \text{ iff } s_1 = s_2 \text{ or } s_1 \text{ and } s_2 \text{ differ only in one or more of their first three bits.}$$

i) Show that $K$ is an equivalence relation

Model answer and marking scheme

- $K$ is reflexive since, $\forall s \in S$, $(s, s) \in K$
- $K$ is symmetric since, $\forall s_1, s_2 \in S$, $(s_1, s_2) \in K \Rightarrow (s_2, s_1) \in K$
- $K$ is transitive since, $\forall s_1, s_2, s_3 \in S$, $(s_1, s_2) \in K \land (s_2, s_3) \in K \Rightarrow (s_1, s_3) \in K$

(2 marks)

ii) Describe the equivalence classes of $K$. 

Page 8 of 10
Model answer and marking scheme

The equivalence classes of $K$ each consist of 8 members, each equivalence class is of the form

$$\{000s, \; 001s, \; 010s, \; 011s, \; 100s, \; 101s, \; 110s, \; 111s\}$$

where $s$ is any bit string. (2 marks)

(4 marks)

Marker’s feedback

Many students showed that relation is an equivalence relation, but very few correctly identified the equivalence classes.
d) i) What is a *partial order*?

**Model answer and marking scheme**
A partial order is a relation which is reflexive, anti-symmetric and transitive. (1 mark)

ii) Explain the terms *least*, and *minimal* with respect to a partial order. Illustrate the difference between these terms with suitable examples.

**Model answer and marking scheme**
An element \( l \) is a least member for a partial order \( \leq \) on a set \( A \) iff,
\[
\forall a \in A, \ l \leq a. \quad (1 \text{ mark})
\]
An element \( m \) is a minimal member for a partial order \( \leq \) on a set \( A \) iff,
\[
\forall a \in A, \ a \leq m \Rightarrow a = m \quad (1 \text{ mark})
\]
Suitable example. (1 mark)

(4 marks)

**Marker’s feedback**
Straightforward bookwork.