UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Mathematical Techniques for Computer Science

18th January 2011

Time: Examination time not specified

Marker’s feedback version
Marking Scheme Included

Please answer any THREE Questions from the FOUR questions provided.

Use a SEPARATE answerbook for each SECTION.

For full marks your answers should be concise as well as accurate.

Marks will be awarded for reasoning and method as well as being correct.

The use of electronic calculators is permitted provided they are not programmable and
do not store text.
1. a) Consider the propositional logic formula $A$ defined as
\[(p \Rightarrow (q \land r)) \land ((p \land q) \Rightarrow r)\]

i) Check if $A$ is satisfiable using the DNF test. (3 marks)

Model answer and marking scheme
\[
(p \Rightarrow (q \land r)) \land ((p \land q) \Rightarrow r)
= (\neg p \lor (q \land r)) \land (\neg (p \land q) \lor r)
= (\neg p \lor (q \land r)) \land (\neg p \lor \neg q) \lor r
= ((\neg p \lor q) \land (\neg p \lor r)) \land ((\neg p \lor \neg q) \lor r)
= (\neg p \lor q) \land (\neg p \lor r) \lor (\neg p \lor \neg q \lor r)
= (\neg p \lor q) \land (\neg p \lor r) \land (\neg p \lor \neg q \lor r)
= (\neg p \lor (q \land r)) \quad \text{CNF}
= (\neg p \lor (q \land r)) \quad \text{DNF}
\]

A is not a contradiction, so is satisfiable.

ii) If so, check if $A$ is a tautology using the CNF test. (2 marks)

Model Answer: $A$ is not a tautology
Distribution of Marks: 2 marks

b) Consider the statement
All Computer Science students use Moodle

Let us define the predicates
\[
CS(x) \quad x \text{ is a Computer Science student}
M(x) \quad x \text{ uses Moodle}
\]

i) Write this statement using the above predicates and quantifiers. (2 marks)

Model Answer: $\forall x \quad (CS(x) \Rightarrow M(x))$
Distribution of Marks: 2 marks

Translate the following statements into English.
Marker’s feedback
Most students got this right but most students didn’t use a proper English sentence for (v) although one could deduce what they meant to say. Many students got (vi) wrong because they started with the expression in (v) and tried to make it into the negation of (ii) instead of doing it the other way around.

ii) \( \exists x \ (CS(x) \land M(x)) \)  

**Model Answer:** Some computer science students use Moodle (or) At least one computer science student uses Moodle  
**Distribution of Marks:** 1 mark

iii) \( \forall x \ (M(x) \Rightarrow CS(x)) \)  

**Model Answer:** All those that use Moodle are Computer Science students  
**Distribution of Marks:** 1 mark

iv) \( \exists x \ (CS(x) \land \neg M(x)) \)  

**Model Answer:** Not all Computer Science students use Moodle (or) At least one Computer Science student does not use Moodle  
**Distribution of Marks:** 1 mark

v) \( \forall x \ (CS(x) \Rightarrow \neg M(x)) \)  

**Model Answer:** No Computer Science student uses Moodle  
**Distribution of Marks:** 1 mark

vi) Show that the expression in (v) is the negation of the expression in (ii)  
(3 marks)

**Model answer and marking scheme**
\[
\neg \exists x \ CS(x) \land M(x) \\
= \forall x \ \neg (CS(x) \land M(x)) \\
= \forall x \ (\neg (CS(x) \lor \neg M(x))) \\
= \forall x \ CS(x) \Rightarrow \neg M(x) \\
\]

c) Let us define a new connective, \( \odot \), whose truth scheme is

\( A \odot B \) is true iff \( A \) and \( B \) have the same truth valuation
(In logic, this connective is known as the XNOR connective)

Marker’s feedback
Most students got (i) (ii) and (iii) right, but about half got (iv) wrong, even though similar examples have been done in class.

i) What is the truth table of the ⊗ connective? (1 mark)

Model answer and marking scheme

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<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A ⊗ B</th>
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<tbody>
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ii) By writing the DNF from the truth table, express ⊗ in terms of ¬, ∨, ∧. (1 mark)

Model answer and marking scheme

\[ A \odot B = (A \land B) \lor (\neg A \land \neg B) \]

iii) Is the connective ⊗ commutative? (2 marks)

Model answer and marking scheme

Yes, from previous part, because \( \land \) and \( \lor \) are. Or from truth table

iv) Is it associative? (2 marks)

(Hint: Use a truth table)

Model answer and marking scheme

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<th>C</th>
<th>B ⊗ C</th>
<th>A ⊗ (B ⊗ C)</th>
<th>A ⊗ B</th>
<th>(A ⊗ B) ⊗ C</th>
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⊙ associative because \( A \odot (B \odot C) \equiv (A \odot B) \odot C \)
2.

Marker’s feedback
This was answered by almost all students.

a) Explain what is meant by the term *formal language*, and name two such languages. (4 marks)

Model answer and marking scheme
A formal language consists of

- A collection of symbols
- A collection of rules to combine these symbols

(2 marks)

Examples of formal languages include Propositional Logic and Java. (2 marks)

Marker’s feedback
A simple piece of book work, generally well answered. Maybe 4 marks was a bit too generous.

b) A small formal language $R$ of expressions is defined by the rules:-

- The lower case letters $a, b, \ldots, z$ are expressions.
- If $A$ is an expression $!A$ is an expression.
- If $A$ and $B$ are expressions, then so are $\oplus AB$, $\ominus AB$ and $\odot AB$

i) Give three examples of expressions in this language, giving the parse tree for each. (3 marks)

Model answer and marking scheme
Expressions include

$\oplus ab$

$! \oplus ab$

$\ominus c! \odot ab$

plus parse trees, each 1 mark

(3 marks)

Marker’s feedback
No great problems here except that some students gave expressions which included upper case letters and brackets, which do not occur in the language, for which they lost marks.
ii) Show that the following is an expression in $R$ by giving its parse tree:

\[ \textcircled{a} \textcircled{y} \textcircled{!} \textcircled{ab} \textcircled{w} \textcircled{v} \textcircled{p} \]

(6 marks)

Model answer and marking scheme

\[
\begin{array}{c}
\text{Model answer and marking scheme} \\
\text{x} \text{ y} \textcircled{=} \\
\text{⋯} \text{!} \text{ a} \text{ b} \textcircled{=} \text{ w} \text{!} \\
\text{⋯} \text{·} \text{ v} \text{!} \\
\text{⋯} \text{·} \text{ p} \textcircled{=} \\
\end{array}
\]

6 marks for correct answer.

Marker’s feedback

Generally very well answered. Some students lost marks because their answers were so untidy that I could not determine whether they were wrong or right.

c) Let $a, b, c$ be fixed (but unknown) rational numbers. The binary operation $\Box$ is defined on the set $\mathbb{Q}$ of rational numbers by

\[ p \Box q = ap + bq + c \]

Marker’s feedback

Answers for this part were a bit disappointing, particularly as it is very similar to an exercise in the notes.

i) Give all possible combinations of values of $a$, $b$ and $c$ if we know that the operation $\Box$ is commutative? (3 marks)
The operation is commutative if and only if

\[ x \boxdot y = y \boxdot x \]

i.e.

\[ ax + by + c = ay + bx + c \]

holds for all \( x, y \in \mathbb{Q} \). This means

\[ (a - b)x + (b - a)y = 0 \]

for all \( x, y \in \mathbb{Q} \).

Therefore \( a = b \). \( c \) can take any value. (3 marks)

ii) Give all possible combinations of values of \( a, b \) and \( c \) if we know that the operation \( \boxdot \) is associative? (3 marks)

The operation is associative if and only if

\[ (x \boxdot y) \boxdot z = x \boxdot (y \boxdot z) \]

holds for all \( x, y, z \in \mathbb{Q} \). i.e.

\[
\begin{align*}
    a(ax + by + c) + bz + c &= ax + b(ay + bz + c) + c \\
    a^2x + aby + ac + bz + c &= ax + bay + b^2z + bc + c \\
    (a^2 - a)x + (b - b^2)z + ac - bc &= 0
\end{align*}
\]

\[ \therefore \]

\[
\begin{align*}
    a^2 - a &= 0 \\
    b - b^2 &= 0 \\
    ac - bc &= 0
\end{align*}
\]

Therefore

\[
\begin{align*}
    a &= 1 \text{ or } a = 0 \\
    b &= 1 \text{ or } b = 0 \\
    a &= b \text{ or } c = 0
\end{align*}
\]

This gives the following possible combinations

\[
\begin{align*}
    a &= 1, b = 1, c \text{ any value.} \\
    a &= 0, b = 0, c \text{ any value.} \\
    a &= 0, b = 1, c = 0 \\
    a &= 1, b = 0, c = 0
\end{align*}
\]

(3 marks)
Marker’s feedback
Very few answered this completely correctly. Many deduced (wrongly) from \( ac - bc = 0 \) that \( a = b \), missing the possibility that \( c = 0 \).

iii) Give all possible combinations of values of \( a, b \) and \( c \) if we know that the operation \( \square \) is both commutative and associative? (1 mark)

Model answer and marking scheme

\[
\begin{align*}
  a = 1, b = 1, & \quad c \text{ any value.} \\
  a = 0, b = 0, & \quad c \text{ any value.}
\end{align*}
\]

(1 mark)
3. a) A store inspects faulty laptops of two types: A and B, and repairs the laptop when possible.

   Of all laptops inspected, 70% are repaired.
   Of those repaired, 60% are type B.
   Of those not repaired, 90% are of type A.

   i) What is the proportion of type B laptops brought in for repair? [4 marks]

   ii) If a faulty computer is of type B, find the probability it will not be repaired? [3 marks]

   iii) If a faulty computer is of type A, find the probability it will be repaired. [3 marks]

**Marker’s feedback**

Q3 a i) 
Some students misinterpreted "laptops brought in for repair" as "laptops repaired" thereby losing marks.
Model answer and marking scheme for Q3

Let events $R =$ ”repair”, $A =$ ”Type A”, $B =$ ”Type B”.

i) Want $\Pr(B)$ given $\Pr(R) = 0.7$, $\Pr(B|R) = 0.6$ and $\Pr(A|R) = 0.9$

$$\Pr(B) = \Pr(B \cap R) + \Pr(B \cap \bar{R})$$

Now $\Pr(B \cap R) = \Pr(B|R) \Pr(R) = 0.6 \times 0.7 = 0.42$

and $\Pr(B \cap \bar{R}) = \Pr(B|\bar{R}) \Pr(\bar{R}) = (1 - \Pr(A|\bar{R})) \Pr(\bar{R}) = 0.1 \times 0.3 = 0.03$

Hence $\Pr(B) = 0.45$

ii) $\Pr(\bar{R}|B) = 1 - \Pr(R|B) = 1 - \frac{\Pr(B \cap R)}{\Pr(B)} = 1 - \frac{0.42}{0.45} = 0.067$

iii) 

$$\Pr(R|A) = \frac{\Pr(R \cap A)}{\Pr(A)} = \frac{\Pr(A|R) \Pr(R)}{\Pr(A)} = \frac{(1 - \Pr(B|R)) \Pr(R)}{\Pr(A)} = \frac{0.4 \times 0.7}{0.55} = \frac{0.55}{0.51}$$

b) Oil is sold at a service station in 5 litre cans. The number of cans sold in an hour is a random variable $X$ distributed with the probability mass function (p.m.f.)

$$
\begin{array}{c|cccc}
 x & 1 & 2 & 3 & 4 \\
p(x) & 0.2 & 0.4 & 0.3 & 0.1 \\
\end{array}
$$

i) Find the mean $E(X)$ and variance $V(X)$ of the distribution. [6 marks]

ii) There are initially 100 litres of oil in stock. Find the expected number and standard deviation of the number of litres of oil left in stock after 1 hour. [4 marks]
Marker’s feedback

Q3 b ii) Let $Y$ be no. of litres of oil in stock. Observing that $Y = 100 - 5X$ gives immediately

$$E(Y) = 100 - 5E(X)$$
$$SD(Y) = 5 \times SD(X)$$

Model answer and marking scheme

Let $X =$ no. of cans sold per hour

i) Mean:
$$E(X) = \sum xp(x) = (1 \times .2) + (2 \times .4) + (3 \times .3) + (4 \times .1) = 2.3$$
Variance:
$$V(X) = E(X^2) - [E(X)]^2 = 6.1 - (2.3)^2 = 0.81$$

ii) No. of litres remaining is a r.v. $Y = 100 - 5X$.
Hence

$$E(Y) = 100 - 5 \times 2.3 = 88.5 \text{ litres}$$
$$s.d.(Y) = 5 \times s.d.(X)$$
$$= 5 \times 0.9$$
$$= 4.5 \text{ litres}$$
4. a) A semiconductor manufacturer knows that 10% of its chips are faulty. Each faulty chip costs 8 seconds of lost production time.

If a random sample of 6 manufactured chips is taken:

i) Find the mean and standard deviation of total lost production time.

   $$\text{E}(T) = 8np$$
   $$\text{SD}(T) = 8\sqrt{npq}$$

   [2 marks]

ii) Find the probability that I) exactly one chip is faulty, II) at least two chips are faulty.

   [4 marks]

iii) If chips are examined one by one, what is the probability that at most five chips must be inspected to find four chips that are not faulty?

   List the successful outcomes of the relevant sample space.

   [4 marks]

Marker’s feedback
Q4 a i)
The time lost is $$T = 8X$$ where $$X$$ has the binomial distribution $$\text{Bin} \ (n, p)$$. Then

$$\text{E}(T) = 8np$$
$$\text{SD}(T) = 8\sqrt{npq}$$

Q4 a iii)
The required probability has the alternative forms:

$$0.9^5 + 5 (0.9)^4 (0.1) = 0.9^4 + 4 (0.9)^4 (0.1)$$
$$= 0.918$$
### Model answer and marking scheme

i) Number of faulty chips $X$ has binomial distribution with $n = 6$ and $p = 0.1$.

So $X \sim Bin(6, 0.1)$. Mean no. faulty chips is $\mu = np = 0.6$.

S.d. is $\sigma = \sqrt{npq}$. Hence

Mean lost production time $= 8\mu = 4.8$ sec.

S.d. of lost production time $= 8\sigma = 5.9$ sec

\[
\text{Pr}(X = 1) = \binom{6}{1} (0.1)^1 (0.9)^5 = 0.345
\]

\[
\text{Pr}(X \geq 2) = 1 - \text{Pr}(X < 1) = 1 - \text{Pr}(X = 0) + \text{Pr}(X = 1) = 1 - \left( 0.9^6 + 0.345 \right) = 0.114
\]

iii) Possible sequences of 5 chips having 4 good chips are:

01111 10111 11011 11101 11110 11111

(being successful outcomes in sample space of first five chips)

Required probability is $0.9^5 + 5(0.9)^4(0.1) = 0.918$

b) It has been empirically observed that *the leading digit* of numbers from a randomly chosen newspaper article appears with a frequency given by *Benford’s law*, according to which the digits 1, 2, ..., 9 have the following *probability mass function* (p.m.f.):

\[
p(x) = \Pr(1^{st} \text{ digit is } x) = \log_{10} \left( \frac{x+1}{x} \right), \quad x = 1, 2, ..., 9.
\]

[This law forms the basis for some auditing procedures used by certain taxation authorities to detect fraud in financial reporting]

i) State the requirements of a valid p.m.f. and show that the set of probabilities satisfies these requirements. [4 marks]

ii) Sketch the p.m.f. and the *cumulative distribution function* (c.d.f.) corresponding to Benford’s law. [2 marks]

iii) What is the probability that the leading digit of a randomly selected number in a newspaper article is I) at most 3? II) At least 5? [4 marks]
Marker’s feedback
Q4 b i) Most candidates failed to observe that $x + 1 > x \Rightarrow \ln \left( \frac{x+1}{x} \right) > 0$
and verified that $\sum p(x) = 1$ by enumerating all probabilities rather than by the preferred method of calculus.
Model answer and marking scheme

i) Requirements of a p.m.f. are 1) \( p(x) \geq 0 \) and 2) \( \sum x p(x) = 1 \)

Since \( \frac{x+1}{x} \geq 1 \), it follows that \( p(x) = \log_{10} \left( \frac{x+1}{x} \right) \geq 0 \).

Normalization:

\[
\sum_x p(x) = \sum_x \log_{10} \left( \frac{x+1}{x} \right) \\
= \log_{10} \prod_x \left( \frac{x+1}{x} \right) \\
= \log_{10} \frac{2 \times 3 \ldots \times 10}{1 \times 2 \ldots \times 9} \\
= \log_{10} (10) \\
= 1
\]

ii) The p.m.f. has the following shape:

![Histogram](image)

c.d.f. is \( P(x) = \Pr(X \leq x) = \log_{10} (x+1) \) with \( P(9) = 1 \) (concave)

iii)

\[
\Pr(X \leq 3) = p(1) + p(2) + p(3) \\
= \log_{10} (2) + \log_{10} \left( \frac{3}{2} \right) + \log_{10} \left( \frac{4}{3} \right) \\
= \log_{10} (4) \\
= 0.602
\]

\[
\Pr(X \geq 5) = 1 - \left[ p(1) + p(2) + \ldots + p(4) \right] \\
= 1 - \log_{10} (5) \\
= 0.301
\]