Mathematical Techniques for Computer Science

?? June 2012

Time: Examination time not specified

Marker’s feedback version

Marking Scheme Included

Please answer any THREE Questions from the FOUR questions provided.

Use a SEPARATE answerbook for each SECTION.

For full marks your answers should be concise as well as accurate.

Marks will be awarded for reasoning and method as well as being correct.

The use of electronic calculators is permitted provided they are not programmable and do not store text.
Section A

1. a) i) Let \( f : \mathbb{N}^+ \rightarrow \mathbb{N}^+ \) be the recursive function defined by

\[
\begin{align*}
    f(1) &= 5 \\
    f(2) &= 13 \\
    f(n+2) &= 5f(n+1) - 6f(n) \quad \text{for } n \geq 1
\end{align*}
\]

Prove by induction that, for all \( n \geq 1 \)
\( f(n) = 2^n + 3^n \)

(6 marks)

Model answer and marking scheme

Need to expand the proposition to be proved:
Let
\[ P(n) \text{ iff } f(n) = 2^n + 3^n \land f(n+1) = 2^{n+1} + 3^{n+1} \]

We prove by induction that \( P(n); \forall n \geq 1 \)

Base case:-
\[ P(1) = f(1) = 2^1 + 3^1 = 5 \land f(2) = 2^2 + 3^2 = 13 \text{ OK.} \quad (2 \text{ marks}) \]

Induction step:-
Suppose, for some \( n, P(n) \)
\[ P(n+1) \text{ is } f(n+1) = 2^{n+1} + 3^{n+1} \land f(n+2) = 2^{n+2} + 3^{n+2} \]

The first clause follows immediately from \( P(n) \)
\[
\begin{align*}
    f(n+2) &= 5f(n+1) - 6f(n) \\
          &= 5(2^{n+1} + 3^{n+1}) - 6(2^n + 3^n) \\
          &= 5.2^{n+1} + 5.3^{n+1} - 6.2^n - 6.3^n \\
          &= 5.2^{n+1} + 5.3^{n+1} - 3.2^n - 2.3^n \\
          &= 5.2^{n+1} + 5.3^{n+1} - 3.2^n + 2.3^n \\
          &= 2^{n+2} + 3^{n+2}
\end{align*}
\]

(4 marks)

Hence \( P(n+1) \).

Marker’s feedback

Most students didn’t do \( f(2) \) for the base case, only \( f(1) \). many students tried to do \( f(n+1) \) as induction step instead of \( f(n+2) \). I think many got confused between \( P(1) \) and \( f(1) \), and \( P(n+1) \) and \( f(n+1) \).
ii) Prove, by induction that for all \( n \geq 0 \)

\[
 r^0 + r^1 + r^2 + \ldots + r^n = \frac{(1 - r^{n+1})}{(1 - r)}
\]

(6 marks)

**Model answer and marking scheme**

Base case:

\[
 L(0) = 1 \quad R(0) = 1 \quad \text{(1 mark)}
\]

Induction step:

\[
 L(n + 1) = L(n) + r^{n+1} \\
 = R(n) + r^{n+1} \\
 = \frac{(1 - r^{n+1})}{(1 - r)} + r^{n+1} \\
 = \frac{(1 - r^{n+2})}{(1 - r)} \\
 = R(n + 1)
\]

This is true for all real numbers \( r \neq 1 \). (1 mark)

**Marker’s feedback**

Most students got this right. A few struggled with the algebra.

b) i) Determine if each of the following functions \( Z \rightarrow Z \) is injective, surjective, neither or both (bijective).

\[
 f(n) = n^2 - 1 \\
 g(n) = n + 1 \\
 h(n) = 2n + 1
\]

(3 marks)

**Model answer and marking scheme**

f is neither injective nor surjective

\( g \) is injective and surjective (bijective)

h is injective but not surjective (1 mark each) (3 marks)

**Marker’s feedback**

Most got this right.

ii) Consider two functions \( p \) and \( q \) which have the properties \( p \) is surjective and \( p \circ q \) is surjective.

Does it follow the \( q \) must also be surjective? Give reasons for your answer.

(3 marks)
<table>
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<tr>
<th>Model answer and marking scheme</th>
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<tbody>
<tr>
<td>q need not be surjective.</td>
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<tr>
<td>Counterexample</td>
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**Marker’s feedback**
Many tried to use words to explain it rather than give a counterexample. Counterexamples are standard in maths but I think some students are simply not aware of the concept coming in to this course.
c) i) Give an example of two functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ which satisfy the following conditions:
- $f$ is injective
- $g$ is surjective
- $g \circ f$ is the identity function
- Neither $f$ nor $g$ is the identity function.

(2 marks)

Model answer and marking scheme
For example:-

\[
\begin{align*}
    f(x) &= x^2 \\
    g(x) &= \lfloor \sqrt{x} \rfloor
\end{align*}
\]

(2 marks)

Marker’s feedback
About half the class got this right. Some misunderstood the question and gave separate examples for each case, rather than one example that satisfies all 3 criteria.
2. a) For each of the following relations, determine whether it is reflexive, whether it is symmetric, whether it is antisymmetric and whether it is transitive. If the relation is an equivalence relation, describe its equivalence classes.

i) The relation $R$ on the set $X = \{a, b, c, d, e, f\}$ is given by

$$R = \{(a, a), (b, b), (c, a), (c, c), (c, d), (c, e), (c, f), (d, d), (e, d), (e, f), (f, f)\}$$

Model answer and marking scheme

- $R$ is not reflexive since, $(e, e) \notin R$
- $R$ is not symmetric since, $(c, a) \in R \land (a, c) \notin R$
- $R$ is anti-symmetric (see digraph)
- $R$ is transitive since, $\forall x, y, z \in X, (x, y) \in R \land (y, z) \in R \Rightarrow (x, z) \in R$

ii) Define the set $\mathbb{N}^{++}$ to be $\{n \in \mathbb{N} \mid n \geq 2\}$ and the relation $C$ on $\mathbb{N}^{++}$ by

$x C y$ iff $x$ and $y$ have a common factor greater than 1.
Model answer and marking scheme

• C is reflexive

• C is symmetric since, if x and y take have a common factor greater than 1, so do y and x.

• C is not anti-symmetric

• C is not transitive, since, if x and y have a common factor and y and z have a common factor, it may not be the case that x and z have a common factor. The factor shared by x and y might not be the same as that shared by y and z. (e.g. 2 C 3 ∧ 6 C 3 but not 2 C 3)

(2 marks)

Marker’s feedback

Most answered this part quite well although some lost marks for not giving any justification for their answers.
b) Define a relation $K$ on $\mathbb{R}$ by

$$x \ K \ y \text{ if and only if } x - y \in \mathbb{Z}$$

i) Show that $K$ is an equivalence relation

**Model answer and marking scheme**

- $K$ is reflexive since, $\forall x \in \mathbb{R}, \ x - x = 0 \in \mathbb{Z}$
- $K$ is symmetric since, $\forall x_1, x_2 \in \mathbb{R}, \ (x_1 - x_2) \in \mathbb{Z} \Rightarrow (x_2 - x_1) \in \mathbb{Z}$
- $K$ is transitive since,
  $\forall x_1, x_2, x_3 \in \mathbb{R}, \ (x_1 - x_2) \in \mathbb{Z} \land (x_2 - x_3) \in \mathbb{Z} \Rightarrow (x_1 - x_3) \in \mathbb{Z}$

(3 marks)

ii) Describe the equivalence classes of $K$.

**Model answer and marking scheme**

The equivalence classes of $K$ are the sets

$$\{ f + n \mid n \in \mathbb{Z} \}$$

where $f \in \mathbb{R}$ satisfies $0 \leq f < 1$  

(2 marks)

**Marker’s feedback**

Many students showed that relation is an equivalence relation, but very few correctly identified the equivalence classes.

(5 marks)

c) A relation $T$ is defined on the set $\mathbb{N}$ by

$$a \ T \ b \text{ if and only if } b = 1 + a$$

i) Find the reflexive closure of $T$

**Model answer and marking scheme**

The reflexive closure is $T \cup \{(x, x) \mid x \in \mathbb{N}\}$  

(1 mark)

ii) Find the symmetric closure of $T$

**Model answer and marking scheme**

The symmetric closure is $T \cup \{(1 + x, x) \mid x \in \mathbb{N}\}$  

(1 mark)

iii) Find the transitive closure of $T$
<table>
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<td>The transitive closure is &lt;</td>
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(2 marks)

(4 marks)

<table>
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<tr>
<th>Marker’s feedback</th>
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<tr>
<td>Many students gave an answer which suggested that the closure (of whatever sort) consists of just the extra edges that need to be added. This is not the case, the closure is the union of the original relation and the extra edges.</td>
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d) i) What is a partial order?

Model answer and marking scheme
A partial order is a relation which is reflexive, anti-symmetric and transitive.  

(1 mark)

ii) Explain the terms least, and minimal with respect to a partial order. Illustrate the difference between these terms with suitable examples.

Model answer and marking scheme
An element \( l \) is a least member for a partial order \( \leq \) on a set \( A \) iff,  
\[ \forall a \in A, \; l \leq a. \]

An element \( m \) is a minimal member for a partial order \( \leq \) on a set \( A \) iff,  
\[ \forall a \in A, \; a \leq m \Rightarrow a = m \]

Suitable example.  

(2 marks)

Marker’s feedback
Straightforward bookwork.

(2 marks)

iii) Show that the relation \( D \) on \( \mathbb{N}^{++} \) (the natural numbers excluding 0 and 1) defined by

\[ a D b \text{ if and only if } a \text{ is a factor of } b \text{ (i.e } \exists k \in \mathbb{N} \text{ with } b = ka) \]

is a partial order. 
Identify any minimal, least, maximal and greatest elements in \( \mathbb{N}^{++} \) with this order.

Model answer and marking scheme

- \( D \) is reflexive since \( a = 1.a \)
- \( D \) is antisymmetric, since \( b = ka \) and \( a = k'b \) means that \( kk' = 1 \) therefore \( k = k' = 1 \) and \( a = b \)
- \( D \) is transitive, since \( b = ka \) and \( c = k'b \) means \( c = k'ka \).
- no least least element (1 would be if it were there)
- All prime numbers are minimal
- No maximal or greatest

(3 marks)
Marker’s feedback
Section B
The general standard on the vectors and matrices questions was very high, with most students tackling both questions in this section and displaying a highly competent grasp of the material. A few specific causes of difficulty:

- A disappointing number of students (including some otherwise very strong ones!) couldn’t put vectors in homogeneous coordinates, despite the repeated coverage of this throughout the course.
- A small but too-large minority of students don’t know their basic values of sin and cos by heart. You should know the values for at least the angles of 0, 30, 45, 60, 90 degrees. In the event this was not penalised because of the mix-up over whether calculators were allowed in the exam, but normally it would be.
- The final couple of parts of Q4 caused rather more difficulties, with most people getting the right idea but many multiplying the matrices in the wrong order.

3. a) Consider the points \( A = (-2, 3), B = (4, 3) \) and \( C = (0, 1) \) in the plane.

i) Plot the points \( A, B \) and \( C \) on two-dimensional coordinate axes. (2 marks)

**Model Answer:** The student should provide an appropriate diagram.

**Distribution of Marks:** 2

ii) Express these points in homogeneous coordinates. (1 mark)

**Model Answer:** \((-2, 3, 1), (4, 3, 1)\) and \((0, 1, 1)\) respectively.

**Distribution of Marks:** 1

iii) Compute the displacement vectors \( \mathbf{p} = \overrightarrow{AB} \) and \( \mathbf{q} = \overrightarrow{AC} \). (1 mark)

**Model Answer:** \( \mathbf{p} = (4, 3) - (-2, 3) = (6, 0), \)
\( \mathbf{q} = (0, 1) - (-2, 3) = (2, -2). \)

**Distribution of Marks:** 1

iv) Compute the magnitudes of \( \mathbf{p} \) and \( \mathbf{q} \). (1 mark)
v) Compute unit vectors parallel to \( \mathbf{p} \) and \( \mathbf{q} \). (2 marks)

Model Answer: \( \hat{\mathbf{p}} = \frac{1}{6} \mathbf{p} = (1, 0) \) and \( \hat{\mathbf{q}} = \frac{1}{2\sqrt{2}} \mathbf{q} = (\sqrt{2}/2, -\sqrt{2}/2) \) respectively.

Distribution of Marks: 2

vi) Consider the triangle with corners at \( A, B \) and \( C \). By computing an appropriate scalar product, give the interior angle of this triangle at \( A \). (3 marks)

Model Answer: The angle in question (call it \( \theta \)) is the angle between \( \mathbf{p} \) and \( \mathbf{q} \) which is the same as the angle between \( \hat{\mathbf{p}} \) and \( \hat{\mathbf{q}} \). Since these are unit vectors they have magnitude 1, so the two different formulae for the scalar product give

\[
\hat{\mathbf{q}} \cdot \hat{\mathbf{p}} = \cos \theta
\]

Working this out yields \( \cos \theta = 1/\sqrt{2} \) so \( \theta = 45^\circ \).

Distribution of Marks: 3

b) Consider the points \( D = (1, 2, 4) \) and \( E = (1, 5, 6) \) in three-dimensional space.

i) How many distinct lines are there which pass through both \( D \) and \( E \)? Explain your answer. (1 mark)

Model Answer: One, since there is a unique line between any two distinct points in space.

Distribution of Marks: 1

ii) Give a vector equation of a line through \( D \) and \( E \). (Call this line \( L_1 \).) (2 marks)

Model Answer: The displacement vector \( \overrightarrow{DE} \) is \( 3\mathbf{j} + 2\mathbf{k} \) so a suitable equation is

\[
\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + \lambda(3\mathbf{j} + 2\mathbf{k})
\]

Distribution of Marks: 2

iii) Does the point \( F = (5, 8, 11) \) lie on the line \( L_1 \)? Explain your answer. (1 mark)
iv) Consider now the line $L_2$ given by the vector equation
\[ \mathbf{r} = -2\mathbf{i} + 17\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}). \]

Do the lines $L_1$ and $L_2$ meet? If so, at what point? If not, explain why not.

(3 marks)

Model Answer: Equating coefficients gives $1 = \mu - 2$, $2 + 3\lambda = 17 - 2\mu$ and $4 + 2\lambda = 1 + 3\mu$. Solving by elimination gives a solution $\lambda = \mu = 3$. Substituting into either equation yields the point $(1, 11, 10)$, so the lines meet at this point.

Distribution of Marks: 3

v) Consider the plane given by the equation
\[ x + 3y + 2z = 0. \]

Does the line $L_2$ intersect this plane? If so, where? Is the line parallel to the plane? Explain your answer.

(3 marks)

Model Answer: A point on the line will have coordinates of the form $x = \lambda - 2, y = 17 - 2\lambda$ and $z = 1 + 3\lambda$. Substituting into the equation of the plane and simplifying gives:
\[ \lambda + 51 = 0 \]

which has a unique solution $\lambda = -51$. Substituting back into the equation of the line gives that the plane and line intersect at the unique point $(-53, 119, -152)$. The line is not parallel to the plane: if it were outside and parallel then it would not intersect, while if it were inside (and hence parallel) it would intersect in more than one point.

Distribution of Marks: 3
4. a) Consider the following four matrices:

\[ A = \begin{pmatrix} 1 & 2 & 4 \\ 5 & 7 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 4 & -5 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & -2 \\ 6 & 1 & -4 \end{pmatrix}, \quad D = \begin{pmatrix} x & 1 \\ 2 & 1 \end{pmatrix} \]

Find each of the following or, if this is not possible, explain why.

i) \( A + B \)  

**Model Answer:** \( A + B \) is not defined as \( A \) has more columns than \( B \).  
**Distribution of Marks:** 1

ii) \( 3C + A \)  

**Model Answer:** \( 3C + A = \begin{pmatrix} 4 & 8 & -2 \\ 23 & 10 & -3 \end{pmatrix} \)  
**Distribution of Marks:** 1

iii) \( AB \)  

**Model Answer:** \( AB \) is not defined as \( A \) has more columns than \( B \) has rows.  
**Distribution of Marks:** 1

iv) \( BA \)  

**Model Answer:** \( BA = \begin{pmatrix} 11 & 16 & 22 \\ -21 & -27 & -29 \end{pmatrix} \)  
**Distribution of Marks:** 2

Find a real number \( x \) such that \( BD = DB \).  

**Model Answer:** Computing products we have  
\[ BD = \begin{pmatrix} x+4 & 3 \\ 4x-10 & -1 \end{pmatrix}, \quad DB = \begin{pmatrix} x+4 & 2x-5 \\ 6 & -1 \end{pmatrix} \]

The non-trivial criteria for these to be equal are \( 3 = 2x - 5 \) and \( 4x - 10 = 6 \) both of which are satisfied by \( x = 4 \).  
**Distribution of Marks:** 3
b) In this part of the question, we work in three dimensions using homogeneous coordinates. Consider the matrices:

\[
P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -

\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

i) Identify the geometric transformations performed by \( P \) and \( Q \). (2 marks)

**Model Answer:** \( P \) performs a rotation through 45° about the \( x \)-axis, while \( Q \) performs the identity transformation.

**Distribution of Marks:** 2

ii) **Without** a direct calculation, describe the matrix \( P^9 \) and the transformation it performs. (2 marks)

**Model Answer:** \( P^9 \) performs nine consecutive rotations through 45° about the \( x \)-axis, which is equivalent to one rotation through \( 9 \times 45° = 405° \) about the \( x \)-axis. But that is the same as a rotation through 45°, so in fact \( P^9 = P \).

**Distribution of Marks:** 2

iii) Write down a matrix \( R \) to perform a translation through \( \vec{i} + \vec{j} + 3\vec{k} \). (2 marks)

**Model Answer:**

\[
R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

**Distribution of Marks:** 2

iv) Write down the inverse of the matrix \( R \). (2 marks)

**Model Answer:** The inverse matrix is the translation through \(- (\vec{i} + \vec{j} + 3\vec{k})\):

\[
R^{-1} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

**Distribution of Marks:** 2
v) Find a matrix which performs a rotation through 45° about a line through (1,1,3) parallel to the x-axis.

**Model Answer:** The transformation can be achieved by translating the given line to the x-axis (using \( R^{-1} \)), rotating around the x-axis (using \( P \)) and then translating the x-axis back to the given line (using \( R \)). Remembering that matrices compose from right to left, the transformation is thus given by the matrix

\[
RPR^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{-1}{\sqrt{2}} & \sqrt{2} + 1 \\
0 & 0 & \frac{1}{\sqrt{2}} & 3 - 2\sqrt{2} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1
\end{pmatrix}
\]

**Distribution of Marks:** 3

vi) Find the image under this transformation of the point (−1, −1, −1).

**Model Answer:**

\[
RPR^{-1} \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{2} + 1 \\ 3 - 3\sqrt{2} \\ 1 \end{pmatrix}
\]

so the image is the point (−1, \( \sqrt{2} + 1 \), 3 − 3\( \sqrt{2} \)).

**Distribution of Marks:** 1

(12 marks)