Two hours

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Mathematical Techniques for Computer Science

Time:

Please answer all THREE Questions

Use a SEPARATE Answerbook for each SECTION
Note that the last two pages contain inference rules for natural deduction

This is a CLOSED book examination

The use of electronic calculators is permitted provided they are not programmable and do not store text.

Section A

1. a) Give a situation in which you, as a programmer, might be concerned with a function being injective. How might you check that your implementation does indeed give an injective function? (3 marks)

b) Consider the following function:

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto z \cdot z. \end{aligned}$$

i) Is this function injective? Justify your answer. (2 marks)

ii) Is this function surjective? Justify your answer. (5 marks)

c) Consider the binary operation on the set $\{a, b, c\}$ given by the following table.

$\circledast$	a	b	c
a	a	a	c
b	a	b	c
c	a	c	c

i) Is the given operation commutative? Give a reason for your answer. (2 marks)

ii) Is the given operation associative? Give a reason for your answer. (4 marks)

d) Show that if

$$f\colon S \longrightarrow T \qquad \text{and} \qquad g\colon T \longrightarrow U$$

are two bijective functions then so is their composite (4 marks)

$$g \circ f\colon S \longrightarrow U.$$

2. a) A fair coin is tossed three times.

   i) Give a probability space that describes this situation. (2 marks)

   ii) What is the probability that the number of heads thrown is even? (1 mark)

   iii) What is the probability that tails occurs at least once? (1 mark)

b) Bonny and Clyde are playing a game. They put two yellow and four green ribbons into a bag. Without looking inside, each of them reaches into the bag and draws a ribbon.

If the ribbons have the same colour Bonny wins and if they are different, then Clyde wins. Is the game fair, that is, do they both have an equal chance of winning? (2 marks)

c) Consider the following situation. An unknown string is assumed to consist of three symbols, each of which can be 0 or 1. It is known that each string has exactly two 0s.

It is only possible to make unreliable queries of the first, second, or third symbol of the string; these questions will return the correct answer with a probability of 0.8.

Assume you start from a probability distribution where all possible strings have the same probability. You are to carry out Bayesian updating based on the following:

- The first symbol is queried and the answer is 0.
- Afterwards the second symbol is queried and the answer is 1.

Explain your calculations. Give your probabilities in fractions. (10 marks)

d) Assume you have a random variable X that takes its values in the set

$$\{0, 1, 2, \ldots, 2n\}.$$

Further assume you know that

$$P(X = n - i) = P(X = n + i) \qquad \text{for } 1 \leq i \leq n.$$

Calculate the expected value of X. (4 marks)

Section B

3. a) Consider the following propositional formula.

$$(Q \rightarrow P) \rightarrow (\neg P \wedge \neg Q)$$

i) Construct a truth table for the formula. Determine if the formula is a tautology. Explain your answer. (3 marks)

ii) Use the CNF algorithm to compute a conjunctive normal form for the formula. Explain all the steps in your calculations. (2 marks)

iii) Simplify the formula obtained in the previous step as much as possible. Explain all the steps in your calculations. (2 marks)

b) Give brief explanations of **two** of the following. (4 marks)

i) connective

ii) contradiction

iii) substitution for a propositional variable

iv) replacement

c) Give a natural deduction proof for the following. Justify every step in your proof.

Note that the inference rules for natural deduction are given on the last pages of this exam paper. (3 marks)

$$P \rightarrow Q, P \rightarrow R \vdash P \rightarrow (Q \wedge R)$$

d) Consider the first-order language with the three binary predicate symbols T, F, E, one unary function symbol t, no constants and a supply of variables $x, y, z, \ldots$.

Suppose the domain of interpretation is the set of first year students and academic staff in the School of Computer Science. Further assume the given predicate and function symbols have the following interpretation:

$T(x,y)$	x and y are in the same tutorial group
$L(x,y)$	x and y are in the same lab group
$F(x,y)$	x is a friend of y
$E(x,y)$	$x = y$
$t(x)$	personal tutor of x

State for each of the following formulas what their interpretation is in the interpretation defined above. (4 marks)

i) $\forall x.\forall y.(T(x,y) \rightarrow E(t(x),t(y)))$
ii) $\forall x.\exists y.(L(x,y) \wedge \neg E(t(x),t(y)))$
iii) $\forall x.\forall y.\big(\exists z.(T(x,z) \wedge L(z,y)) \rightarrow T(x,y)\big)$

Using the language and interpretation given above, express each of the following sentences as first-order formulas. (2 marks)

iv) Everyone in a tutorial group has a friend in the tutorial group.
v) No one in a tutorial group is friends with everyone in the tutorial group.

Rules of inference of our propositional natural deduction system

Conjunction elimination:

If $A \wedge B$ is derivable from a set of formulas, then so is A, and also B.

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

Conjunction introduction:

If A is derivable from a set of formulas, and B is derivable from the same set, then $A \wedge B$ is derivable from this set as well.

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

Disjunction introduction:

If A is derivable from a set, then so is $A \vee B$, and also $B \vee A$.

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash B \vee A}$$

Disjunction elimination (proof by cases):

If $A \vee B$ is derivable from a set and C is derivable from the set along with A, and also from the set along with B, then C is derivable from the set alone.

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

Implication introduction:

If B is derivable from A and a set, then $A \to B$ is derivable from the set .

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

Implication elimination:

If A is derivable from a set, and $A \to B$ is derivable from the same set, then B is derivable from this set.

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash A \to B}{\Gamma \vdash B}$$

Negation introduction (reductio ad absurdum):

If A and a set leads to a contradiction, then $\neg A$ can be inferred from the set.

$$\frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}$$

Negation elimination:

If B is derivable from a set, and also $\neg A$ is derivable from the set, then anything (including $\bot$) is derivable from the set.

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash \neg A}{\Gamma \vdash B}$$

Double negation introduction:

If A is derivable from a set, then $\neg\neg A$ is derivable from the same set.

$$\frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A}$$

Double negation elimination:

If $\neg\neg A$ is derivable from a set, then A is derivable from the same set.

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

Axiom (starting point):

A can always be inferred from A and a set of formulas.

$$\frac{}{\Gamma, A \vdash A}$$

Weakening:

New assumptions may be introduced at any point in a derivation.

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$

END OF EXAMINATION